

CONSEQUENCES OF THE FIXED POINT LEMMA

FPL: For any formula $B(v)$ in the language of H , with exactly v free, there is a sentence F such that $\vdash_H \ulcorner F \equiv B(\ulcorner F \urcorner) \urcorner$.

Tarski's Indefinability Theorem

TT: The set of Gödel numbers of truths in \mathcal{N} , call it $\text{Tr}(\mathcal{N})$ is not arithmetically definable in H .

Definition. A set S of natural numbers is **arithmetically definable** in the language of arithmetic iff there is a formula $A(v)$ of that language with exactly v free such that:

$$n \in S \text{ iff } \mathcal{N} \models \ulcorner A(\underline{n}) \urcorner$$

The Basic Argument

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| (0) $\mathcal{N} \models A$ iff $\ulcorner A \urcorner \in \text{Tr}(\mathcal{N})$ | [By definition of $\text{Tr}(\mathcal{N})$] |
| (1) $T(v)$ is such that $\ulcorner A \urcorner \in \text{Tr}(\mathcal{N})$ iff $\mathcal{N} \models \ulcorner T(\ulcorner A \urcorner) \urcorner$ | [Suppose for <i>reductio</i>] |
| (2) $\vdash_H \ulcorner L \equiv \sim T(\ulcorner L \urcorner) \urcorner$ | [From FPL] |
| (3) $\mathcal{N} \models \ulcorner L \equiv \sim T(\ulcorner L \urcorner) \urcorner$ | [From (2) assuming $\mathcal{N} \models H$] |
| (4) $\mathcal{N} \models L$ iff $\mathcal{N} \not\models \ulcorner T(\ulcorner L \urcorner) \urcorner$ | [From (3)] |
| (5) $\mathcal{N} \not\models L$ iff $\mathcal{N} \not\models \ulcorner T(\ulcorner L \urcorner) \urcorner$ | [From (0) and (1)] |
| (6) $\mathcal{N} \models L$ iff $\mathcal{N} \not\models L$ | [From (4) and (5)] |
| (7) No $T(v)$ arithmetically defines $\text{Tr}(\mathcal{N})$ | [By <i>reductio</i> ; contradiction at (6)] |

The Undecidability of Arithmetic (Revisited)

UA: The set of Gödel numbers of theorems in H , call it $\text{Th}(H)$, is not decidable. This is immediate from the following two premises:

SRT: Any decidable set is strongly represented in H . [Strong Representation Theorem]¹
Lemma: $\text{Th}(H)$ is not strongly represented in H .

Proof of Lemma:

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| (0) $\vdash_H A$ iff $\ulcorner A \urcorner \in \text{Th}(H)$ | [By definition of $\text{Th}(H)$] |
| (1) $P(v)$ is such that $\ulcorner A \urcorner \in \text{Th}(H)$ iff $\vdash_H \ulcorner P(\ulcorner A \urcorner) \urcorner$,
and $\ulcorner A \urcorner \notin \text{Th}(H)$ iff $\vdash_H \ulcorner \sim P(\ulcorner A \urcorner) \urcorner$ | [Suppose for <i>reductio</i>] |
| (2) $\vdash_H \ulcorner G \equiv \sim P(\ulcorner G \urcorner) \urcorner$ | [From FPL] |
| (3) $\vdash_H G$ iff $\vdash_H \ulcorner \sim P(\ulcorner G \urcorner) \urcorner$ | [From (2)] |
| (4) $\not\vdash_H G$ iff $\vdash_H \ulcorner \sim P(\ulcorner G \urcorner) \urcorner$ | [From (0) and 2 nd conjunct of (1)] |
| (5) $\vdash_H G$ iff $\not\vdash_H G$ | [From (3) and (4)] |
| (6) No $P(v)$ strongly represents $\text{Th}(H)$ | [By <i>reductio</i> ; contradiction at (5)] |

Interestingly, the extension of the proof-predicate ' $\text{Pf}_H(n, m)$ ' is strongly represented in H , but (6) means that the extension of the existential-proof-predicate ' $\exists x \text{Pf}_H(x, m)$ ' is *not*.

¹ This follows from EL and the fact that the characteristic function for any decidable set is recursive.