CONSEQUENCES OF THE FIXED POINT LEMMA

FPL: For any formula B(v) in the language of H, with exactly v free, there is a sentence F such that $\vdash_H \ulcorner F \equiv B(\underline{^\#F})\urcorner$.

Tarski's Indefinability Theorem

TT: The set of Gödel numbers of truths in $\mathcal{N}$, call it Tr($\mathcal{N}$) is not arithmetically definable in H.

Definition. A set S of natural numbers is **arithmetically definable** in the language of arithmetic iff there is a formula A(v) of that language with exactly v free such that:

$n \in$ S iff $\mathcal{N} \vDash \ulcorner A(\underline{n})\urcorner$

The Basic Argument

(0) $\mathcal{N} \vDash$ A iff $^\#A \in$ Tr($\mathcal{N}$)	[By definition of Tr($\mathcal{N}$)]
(1) T(v) is such that $^\#A \in$ Tr($\mathcal{N}$) iff $\mathcal{N} \vDash \ulcorner T(\underline{^\#A})\urcorner$	[Suppose for *reductio*]
(2) $\vdash_H \ulcorner L \equiv \sim T(\underline{^\#L})\urcorner$	[From FPL]
(3) $\mathcal{N} \vDash \ulcorner L \equiv \sim T(\underline{^\#L})\urcorner$	[From (2) assuming $\mathcal{N} \vDash$ H]
(4) $\mathcal{N} \vDash$ L iff $\mathcal{N} \nvDash \ulcorner T(\underline{^\#L})\urcorner$	[From (3)]
(5) $\mathcal{N} \nvDash$ L iff $\mathcal{N} \nvDash \ulcorner T(\underline{^\#L})\urcorner$	[From (0) and (1)]
(6) $\mathcal{N} \vDash$ L iff $\mathcal{N} \nvDash$ L	[From (4) and (5)]
(7) No C(v) arithmetically defines Tr($\mathcal{N}$)	[By *reductio*; contradiction at (6)]

The Undecidability of Arithmetic (Revisited)

UA: The set of Gödel numbers of theorems in H, call it Th(H), is not decidable.

Premises:
SRT: Any decidable set is strongly represented in H. [Strong Representation Theorem][1]
Lemma: Th(H) is not strongly represented in H.

Proof of Lemma:

(0) $\vdash_H$ A iff $^\#A \in$ Th(H)	[By definition of Th(H)]
(1) P(v) is such that $^\#A \in$ Th(H) iff $\vdash_H \ulcorner P(\underline{^\#A})\urcorner$, and $^\#A \notin$ Th(H) iff $\vdash_H \ulcorner \sim P(\underline{^\#A})\urcorner$	[Suppose for *reductio*]
(2) $\vdash_H \ulcorner G \equiv \sim P(\underline{^\#G})\urcorner$	[From FPL]
(3) $\vdash_H$ G iff $\vdash_H \ulcorner \sim P(\underline{^\#G})\urcorner$	[From (2)]
(4) $\nvdash_H$ G iff $\vdash_H \ulcorner \sim P(\underline{^\#G})\urcorner$	[From (0) and 2nd conjunct of (1)]
(5) $\vdash_H$ G iff $\nvdash_H$ G	[From (3) and (4)]
(6) No P(v) strongly represents Th(H)	[By *reductio*; contradiction at (5)]

Interestingly, the extension of the proof-predicate '$Pf_H(n, m)$' *is* strongly represented in H, but (6) means that the extension of the existential-proof-predicate '$\bigvee x\ Pf_H(x, m)$' is *not*.

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[1] This follows from EL and the fact that the characteristic function for any decidable set is recursive.