FUNCTIONS AND RELATIONS

1. Functions

Intuitively, a **function** F is an operation or rule which takes something as input and associates it with a unique output. You're already familiar with functions: Addition is a function which takes a pair of numbers as input, and outputs their sum. Squaring is a function that takes a single number as input, and outputs the number multiplied by itself. Etc. Take note that an "input" or "output" can be an *n*-tuple, e.g., addition officially takes an ordered pair as input.

The set of inputs to a function F is the **domain** of F, written 'domF', whereas the set of outputs is its **range**, written 'ranF'.[1] A function satisfies **well-definedness**: Each member of domF is mapped to one and only one member of ranF.

Division is still a function even though 1/0 yields no output. It just means that $\langle 1, 0\rangle$ is not in its domain. Similarly, the anti-successor function on $\mathbb{N}$ does not have 0 in its domain. Yet 0 is in the domain of the anti-successor function on $\mathbb{Z}$, for that function outputs -1 when applied to 0. Thus, terms like 'the anti-successor function' are a domain-relative, although in the context of use, it is often unnecessary to say explicitly which domain is meant.

As the above illustrates, we say that a function F is **on a set A** if domF $\subseteq$ A and ranF $\subseteq$ A.[2] When domF $\subset$ A, F is a **partial function on A**, as with the anti-successor function on $\mathbb{N}$. Whereas, if domF = A, F is **total function on A**, e.g., the successor function on $\mathbb{N}$.

Notation: If d is an input to a function F, then 'F(d)' denotes the output of F on that input; e.g., if the successor function on $\mathbb{N}$ is $S(x) = x + 1$, then 'S(2)' denotes 3. Such a term can then indicate the input in further cases: Where $G(x) = x^2$, $G(S(2)) = 9$, $G(G(S(2))) = 81$, etc. This extends to *n*-ary functions generally, although angle brackets are dropped. Instead of '$F(\langle 1, 2\rangle)$', we write: F(1, 2).

Technically, a function is defined *not* by specifying an operation or rule, but rather just by a set of input-output pairs of the form "$\langle$input, output$\rangle$"

E.g., the squaring function on $\mathbb{Z}$ is $\{\ldots\langle -2, 4\rangle, \langle -1, 1\rangle, \langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 4\rangle\ldots\}$, and the addition function on $\mathbb{P}$ is $\{\langle\langle 1, 1\rangle, 2\rangle, \langle\langle 1, 2\rangle, 3\rangle\ldots\langle\langle 2, 1\rangle, 3\rangle, \langle\langle 2, 2\rangle, 4\rangle\ldots\}$.

This means that functions F and G are identical iff they have the same outputs on the same inputs (even if the operations/rules that characterize F vs. G are very different). This illustrates how functions are really just part of set theory.

If domF = A and ranF $\subseteq$ B, then F maps A **into** B, written 'F: A $\rightarrow$ B'. If ranF = B, we can say more precisely that F maps A **onto** B. Note that a function "on A" might not be *onto* A, e.g., the squaring function on $\mathbb{Z}$. (This is one reason why 'over A' is sometimes used instead of 'on A'.)

A function F is **one-to-one** iff it never yields the same output on two different inputs. This is the converse of well-definedness.

I.e., F is one-to-one iff, for all x and y, $F(x) = F(y)$ only if $x = y$.

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[1] Some call the range the *image* or *co-domain*. (And if that set is a proper subset of B, they may instead call B the "range," but let's ignore that.)

[2] But not vice-versa, it seems. Consider that addition "on $\mathbb{N}$" does not have members of $\mathbb{N}$ in its domain, but rather members of $\mathbb{N} \times \mathbb{N}$. So apparently, mathematicians say a function is "on A" iff the function has as inputs members of A *or* members of some Cartesian power of A. (Ditto with relations.)

Iff some one-to-one function maps A *onto* B, then there is a **1-1 correspondence between A and B**.[3] This is equivalent to the definition from the set theory handout, but the present definition is the one that is usually appealed to.

2. Binary and Other Relations

A **binary relation** on sets A and B is a set of ordered pairs $\langle x, y\rangle$ where $x \in$ A and $y \in$ B. We speak here too of A as the domain and B as the range. Indeed, all functions are binary relations, but not vice-versa: Some binary relations pair an element of the domain with more than one element of the range. E.g., the relation $x < y$ on $\mathbb{Z}$ pairs any integer to infinitely many other integers. In the case of $x = 1$, the relation has as members $\langle 1, 2\rangle$, $\langle 1, 3\rangle$, $\langle 1, 4\rangle$, etc.

The **inverse** of a binary relation switches the order of each pair in the relation. Thus, the inverse of $x < y$ is $x > y$ since the latter has as members $\langle 2, 1\rangle$, $\langle 3, 1\rangle$, $\langle 4, 1\rangle$, etc. If the relation is a one-to-one function F, then its inverse is often written as the function F^{-1}. But the inverse of F is itself a function *only if* F is one-to-one.

Generalizing, if $A_1, \ldots, A_n$ are sets, then an ***n*-ary relation R on $A_1, \ldots, A_n$** is a set of n-tuples $\langle x_1, \ldots, x_n\rangle$ such that $x_1 \in A_1$, and …, and $x_n \in A_n$. (We also say that the relation has an **arity** of n.) But almost always, we are concerned with n-ary relations defined entirely on a chosen set A, meaning that the n-tuples $\langle x_1, \ldots, x_n\rangle \in$ R have elements all from the same set A. (In fact, we often just consider functions and other binary relations on A.)

As with functions, relations in general are technically defined by sets of n-tuples whose members are in the relevant relation. Thus, the relation $x = y$ on $\mathbb{N}$ just is the set $\{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle, \ldots\}$

Properties of Binary Relations Where 'x' 'y' and 'z' vary over elements of domR:

Reflexive: For any x, Rxx.
  Examples: $x = y$ and $x \leq y$.
Irreflexive: For any x, ~Rxx.
  Example: $x + 1 = y$.
  Irreflexive is stronger than non-reflexive. E.g,, $x = y^2$ is non-reflexive but not irreflexive.
Symmetric**:** For any x and y, Rxy only if Ryx.
  Examples: $x = y$ and $x \neq y$.
Asymmetric: For any x and y, Rxy only if ~Ryx. (N.B., Asymmetric implies irreflexive.)
  Examples: $x < y$ and $A \subset B$.
  Asymmetric is stronger than non-symmetric. E.g., $x \leq y$ is non-symmetric but not asymmetric.
Anti-symmetric: For any x and y, if Rxy and Ryx, then $x = y$.
  Examples: $x = y^2$, $x \leq y$, and $A \subseteq B$.
  -The term here is misleading; symmetry is independent of anti-symmetry. E.g., $x = y$ is symmetric and anti-symmetric. Though asymmetry vacuously implies anti-symmetry: No asymmetric relation satisfies the antecedent in the definition of anti-symmetric.
Transitive: For any x, y, and z, if Rxy and Ryz, then Rxz.
  Examples: $x < y$, $x \leq y$, $A \subset B$, and $A \subseteq B$. Whereas, $A \in B$ and $x \neq y$ are non-transitive.

[3] Lots of people use the terms 'injection', 'surjection', and 'bijection' in this area, but I find that terminology confusing for more than one reason. I try to avoid it.

A binary relation R is an **equivalence relation** iff R is reflexive, symmetric, and transitive. The most prominent example is $x = y$, but there are many others, e.g., A $\asymp$ B. The above list is mainly a way to get to equivalence relations—and that gets us to the most important concept in this area:

If R is an equivalence relation, then for any x, the **equivalence class of *x*** is $\{y : \mathrm{R}yx\}$, often written as $[x]$. It is the set of things that are "equivalent" to x under the relation R.

<u>Application 1</u>: ['G(2)'] under the relation $\equiv$ is the set of formulae equivalent to 'G(2)', whose elements include formulae like '~~G(2)' and 'G(2) $\vee$ G(2)'. Importantly, this equivalence class can effectively stand for the unique *proposition* expressed by these formulae.

<u>Application 2</u>: For $n, m \in \mathbb{P}$, a negative integer z might be seen as the ordered pair $\langle n, m\rangle$ if $n < m$ and $n - m = z$. However, 1 – 2 and 2 – 3 both equal -1, even though $\langle 1, 2\rangle \neq \langle 2, 3\rangle$. Yet we can instead define -1 as the equivalence class of pairs $\langle x, y\rangle$ under the relation $x = y + 1$ where $x \in \mathbb{N}$. This equivalence class has as its elements $\langle 1, 2\rangle$, $\langle 2, 3\rangle$, $\langle 3, 4\rangle$, etc., which are exactly what we want. Moreover, $[\langle 1, 2\rangle] = [\langle 2, 3\rangle]$, confirming that $[\langle 1, 2\rangle]$ is apt as a definition of -1. Hence, negative integers are often identified as the equivalence classes $[\langle x, y\rangle]$ under $<$ on $\mathbb{P}$. (An analogous use of equivalence classes occurs when defining the rational numbers.)

As these applications illustrate, talk of equivalence classes are ways of talking about separate "chunks" or **partitions** of the domain where the individuals in each partition are indiscernible under the relevant relation. When we prove things like completeness, dividing up formulae this way makes the task more manageable.

Equivalence relations are great, but transitive, non-equivalence relations deserve attention too. These often define an *order* on the domain, where we can speak of (at least some) objects being "earlier" or "later" than (at least some) objects. Here are some of the main types of orders:

<u>Partial order</u>: R is transitive, reflexive, and anti-symmetric.
  Examples: A $\subseteq$ B and $x \leq y$.[4]
<u>Linear or Total order</u>: A partial order with *totality*: For any x and y, if $x \neq y$ then Rxy or Ryx.
  Example: A $\subseteq$ B in ZFC, but not necessarily in ZF. Totality fails in ZF-models with incomparable cardinals.
<u>Strict order</u>: R is transitive and irreflexive.
  Examples: A $\subset$ B and $x < y$.
<u>Well-ordering</u>: A strict order with totality where every nonempty subset has a least element.[5]
  Example: $x < y$ on $\mathbb{N}$ but not on $\mathbb{Z}$. ($\mathbb{Z}$ is a subset of itself and has no least element.)

This vocab then allows us to tackle the "official" definition of an ordinal:
A is an **ordinal** set/number iff: A is well-ordered by the relation $\in$, and $x \in$ A only if $x \subseteq$ A.

[4] Technically, $x = y$ is a partial "order" albeit a degenerate one.
[5] Mathematicians sometimes talk of a "strict total order," which may sound contradictory. (A strict order is irreflexive, but a "total order" is a partial order, hence, reflexive.) What they mean is a strict order (irreflexive) that also has the property of totality.