CONSEQUENCES OF THE MODEL EXISTENCE LEMMA FOR K

45.20 The Compactness Theorem for K

45.20 If every finite subset of proper axioms from a first order theory K has a model, then K has a model. (Since the other direction is trivial, an 'iff' would be just as well.)

Proof: Suppose for conditional proof that each finite subset of proper axioms from K has a model. And suppose for *reductio* that K has no model. Then, K is inconsistent, by MEL for K. So by the consistency of [K1]-[K7] and the finitude of derivations, there is a finite subset Δ of proper axioms of K such that $\Delta \vdash_K A$, and $\Delta \vdash_K \sim A$, for some wff A. Thus, by the meaning of '~', Δ has no model. So there is a finite subset of proper axioms from K without a model, contra our initial supposition. So by *reductio*, K has a model.

45.18 The Downward Löwenheim-Skolem Theorem

Since any first order theory with a model is consistent, the (Normal/) MEL for K directly implies:

45.18: If a first order theory K has a model, then K has a denumerable (normal/) model.

Skolem's "Paradox" Set theory can be axiomatized as a first order theory $K_{\{\}}$ with '='. After all, we can translate the ZFC axioms into Q, add them to the proper axioms from [$K^{=}1$] and [$K^{=}2$], and then add the logical axioms from [K1]-[K7] to get $K_{\{\}}$. Now if set theory is consistent (?), then by Normal MEL, $K_{\{\}}$has a normal model. Hence, 45.18 implies $K_{\{\}}$has a denumerable normal model M*. But in M*, such a theory "says" that there are uncountably many sets, even though M* is a normal model with only countably many objects.

This initially seems paradoxical but recall that M* interprets set-theoretic vocabulary in a self-referential way. E.g., the extension of 'is uncountable' is the set S of closed terms c such that "c is uncountable" is a theorem of $K_{\{\}}$. And S must be countable since there are only denumerable closed terms of Q or Q+ in total.

49.2 Non-Standard Interpretations of Arithmetic

Henceforth, assume that a "first order theory" does not have propositional symbols.

49.2: Let $K^{=}$ be a consistent first order theory with '='. Then, even if $K^{=}$ has denumerably many proper axioms, $K^{=}$ has normal models where the predicate for 'is a natural number' is not satisfied by any natural number.

Proof: By the Normal MEL, $K^{=}$ has a normal model whose domain is a set of closed terms of Q. Closed terms are not numbers.

Ted's Favorite Theorem [TFT]" If a first order arithmetical theory with '=' (such as Robinson arithmetic) has its intended model, it has an *isomorphic* model where the arithmetical language has a non-standard interpretation (meaning its terms and predicates have different denotations than in the intended model).

Definition (rough). If R is a first order arithmetical theory with '=', its **intended model** is the normal model $\mathcal{N}$ with domain $\mathbb{N}$, where terms denote natural numbers, where predicates like '>' denote the usual properties/relations on such numbers, etc.

Definition (partial): If K has model M, an **isomorphic** model $M^* \neq M$ is a model where all *and only* wff of K that are true in M are true in M*. More specifically…

Definition (complete): A model M with domain $D = \{d_1, d_2, d_3\ldots\}$ for a first order theory K has an **isomorphic** model M* with domain D* iff there is a one-one function $g(d)$ from D onto D* such that:

1. M assigns a constant c to d_k iff M* assigns c to to $g(d_k)$.
2. M assigns a functor f to a function f iff M* assigns f to f^*, where $f(d_1\ldots d_n) = d_k$ iff $f^*(g(d_1)\ldots g(d_n)) = g(d_k)$.
3. M assigns a predicate F to a relation R iff M* assigns F to R*, where $\langle d_1\ldots d_n\rangle \in R$ iff $\langle g(d_1)\ldots g(d_n)\rangle \in R^*$.

Observe that if K contains '=' and has a model, then '=' will express identity in any isomorphic model. After all, since $g(d)$ is a function, the pairs of identicals in D will get mapped to pairs of identicals in D*, as per clause 3 above. So if K contains '=', then all isomorphisms for K are normal models for K.

Proof of TFT: Left as an exercise.

Remark: TFT can be generalized to show that *any* consistent first order theory has an isomorphism featuring a non-standard interpretation. (Cf. Putnam, "Models and Reality".)

<u>48.4: Non-Standard Models of Arithmetic</u>

We just saw how consistent arithmetical theories have isomorphic models. But such theories also have non-isomorphic models.

48.4: If a first order arithmetical theory with '=' has its intended model $\mathcal{N}$, then it also has a normal model that is not isomorphic to $\mathcal{N}$ (i.e., the theory has a "non-standard model").

Proof: Take a consistent arithmetical theory R and expand it by including a new constant 'c', and by adding denumerably many proper axioms of the form "$\underline{n} \neq c$," for each numeral $\underline{n}$. Call this expanded theory R*. Every finite subset of the new proper axioms has a model: For a given finite set of such axioms, just let c name the highest number from the standard model that is not named on the left-hand side of those axioms. Hence, by Compactness (and by the fact that axioms from [K1]-[K7] are logically valid), it follows that R* has a model.

Since this shows that R* is consistent, then by the Normal MEL, it has a normal model M*. But M* is not isomorphic with the intended model $\mathcal{N}$. If it were, then the truth of all the axioms of the form "$\underline{n} \neq$ c" means there would be a number that is non-identical to every number. Since that is impossible, M* has something distinct from every number, hence, M* is non-isomorphic to $\mathcal{N}$. But since it is a model for R*, and R* is an extension of R, then it is a model for R. So R has a non-standard model.

Remark: In the first instance, this suggests that a consistent arithmetical theory has a model with a non-standard *domain*, thanks to M* containing a non-number. Yet all arithmetical truths remain true in M*, including $\bigwedge x' x' \geq 0$. This means '$x' \geq 0$' is satisfied by a non-number in M*. Thus, M* has a non-standard interpretation of some arithmetical vocabulary, besides having non-standard domain.

The Upward Löwenheim-Skolem Theorem

This tactic of introducing κ new constants along with the correlative inequality axioms is also (in brief) how you prove the Upward Löwenheim-Skolem Theorem:

ULS: If a first order theory K has a denumerable model, then it has a model of arbitrary infinite cardinality κ.