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Naive Sets and Russell's Paradox

1.1 Sets

Here is how philosophers and mathematicians think of sets. If you have some things—people, cars, trees, numbers, countries, any sort of things—then there is also a *further* thing, the *set* containing those things.

So, if we start with Margaret Thatcher, Tony Blair, and Albert Einstein, for example, we then have the set containing these three, namely: {Margaret Thatcher, Tony Blair, Albert Einstein}. Or if we start with London, Jane Austen, the number 3, and Iceland, we then have: {London, Jane Austen, 3, Iceland}.

Similarly, if we start with all the cars in London, we have the set {x: x is a car in London}. (Read this as: the set of xs such that x is a car in London.) Or if we start with all the countries in Europe, we then have {x: x is a country in Europe}.

Note how we can specify a set by listing all its members, as in the first two examples above, or by specifying a property that picks out all its members, as in the second two.

In the former examples, we are using the *extensive notation* for a set. We name the set by naming the members in turn inside squiggly brackets. {Margaret Thatcher, Tony Blair, Albert Einstein}. {London, Jane Austen, 3, Iceland}.

In the latter examples we are using the *intensive notation* for a set. We name the set by specifying a feature common to all its members inside squiggly brackets. {x: x is a car in London}. {x: x is a country in Europe}.

Sometimes we can name a set in both ways: {John, Paul, George, Ringo}, {x: x is a Beatle}. Note that these aren't two different sets, just two different ways of naming the same set.

1.2 Membership and the Axiom of Extensionality

We say that a set *contains* its *members*, and the members *belong* to the set. If S is a set and m belongs to it, we write 'm ∈ S'. ∈ is the *membership relation*.

The nature of a set depends on nothing more than its members. If A and B are sets, then they are the same set if and only if they have the same members. More formally we can write:

For any sets A, B: A = B iff[1] (for any x)(x ∈ A iff x ∈ B).

This principle is known as the *axiom of extensionality*. It makes it explicit when two sets are the same—just in case they have the same members. At the end of the chapter we will meet another axiom—the *axiom of comprehension*—that makes it explicit what sets there are in the first place.

Together these two axioms constitute *naive set theory*.

(An 'axiom' is a basic assumption of a theory. A theory can be viewed as all the statements that follow by logic from its

[1] Philosophers and mathematicians use 'iff' as a handy abbreviation for 'if and only if'.

Box 1 The Reality of Sets

Do sets really exist? Do we really want to allow that in addition to Margaret Thatcher, Tony Blair, and Albert Einstein, there is an *extra* thing, the set {Margaret Thatcher, Tony Blair, Albert Einstein}? Where is this extra thing located? Does it make any difference to anything else? Certainly some philosophers deny the existence of sets, and view them as nothing but useful fictions made up by mathematicians. However, we can put such doubts to one side for present purposes. Think of this chapter as exploring the properties that sets *would* have, *if* they existed. Even if you are sceptical about sets, you will do well to understand what you are objecting to. Know thine enemy.

axioms. We shall look at axioms and theories in more detail in Chapter 12.)

1.3 Unions, Intersections, and the Empty Set

The *union* of sets A and B is the set which contains everything that belongs to *either* A *or* B or both. We write A ∪ B.

So {Margaret Thatcher, Tony Blair, Madonna} ∪ {Jane Austen, Tony Blair, Iceland} = {Margaret Thatcher, Tony Blair, Madonna, Jane Austen, Iceland}.

The *intersection* of sets A and B is the set which contains everything that belongs to *both* A *and* B. We write A ∩ B.

So {Margaret Thatcher, Tony Blair, Madonna} ∩ {Jane Austen, Tony Blair, Iceland} = {Tony Blair}.

There is also an *empty set,* a set which exists but has no members. We write {}, or Ø.

1.4 Subsets

If A is a set, then B is a *subset* of A if and only if all the members of B are also members of A. We write B $\subseteq$ A.

So {Margaret Thatcher, Tony Blair} and {Tony Blair, Jane Austen, Iceland} are both subsets of {Margaret Thatcher, Tony Blair, Madonna, Jane Austen, Iceland}.

The 'singleton set' {Margaret Thatcher} is also a subset of this set. This is the set whose only member is Margaret Thatcher. Be careful not to muddle up this singleton subset with Margaret Thatcher herself. Margaret Thatcher is a person, not a set.

Note that every set is a subset of itself. (We specified above that B is a subset of A if all the members of B are also members of A. Well, given any set A, all the members of A are certainly members of A.)

If B is a subset of A other than A itself we say it is a *proper* subset, and write B $\subset$ A.

The empty set is a subset of every set. (This might seem a bit arbitrary. *Is* every member of the empty set also a member of every other set, in line with the above definition of a subset? Since the empty set doesn't have any members, it is not obvious whether this is true. Still, let us agree to understand the definition of a subset in this way. Things work out more neatly if we count the empty set as a subset of every set.)

1.5 Members versus Subsets

As we saw above, being a member is *not* the same as being a subset. Subsets of A are extra *sets*, each of which contain some members of A, and as such are not normally members of A themselves.

Even so, in certain cases a subset of a set *can* also be a member of that set.

This is possible because sets can have other sets as their members. Remember that sets are things in their own right, and that any things can enter into sets. So sets, along with ordinary objects, can be members of other sets.

To illustrate how one set can be a member of another, suppose we start with the people Elvis Presley and John Lennon, plus the sets {Margaret Thatcher, Tony Blair} and {Albert Einstein, Stephen Hawking}. Then there will be another set which has just those things as members, namely:

{Elvis Presley, John Lennon, {Margaret Thatcher, Tony Blair}, {Albert Einstein, Stephen Hawking}}.

Note how this set has both people and sets as members.

We can now see how it is possible for a subset of a set to be a member of that same set. For example, consider this set A:

{Ringo Starr, Paul McCartney, {Margaret Thatcher, Tony Blair}, {Ringo Starr, Paul McCartney}}.

The set {Ringo Starr, Paul McCartney} is a *member* of A—namely, the last-named member. But it is also a subset of A, because both its members are members of the set A—namely, the first two members of A.

Note that {Ringo Starr, Paul McCartney} is not a member of A *because* it is a subset of A. For it to be a subset, all that is required is that its *members* are members of A. It is a further fact that it is itself a member of A.

To drive the point home, consider this set B:

{Ringo Starr, Paul McCartney, {Margaret Thatcher, Tony Blair}}.

Now {Ringo Starr, Paul McCartney} is a subset of B, but *not* a member of B.

1.6 Power Sets

The set {Ann, Bob} has 4 subsets:

Ø, {Ann}, {Bob}, {Ann, Bob}.

The set {Ann, Bob, Clio} has 8 subsets:

Ø, {Ann}, {Bob}, {Clio}, {Ann, Bob}, {Ann, Clio}, {Bob, Clio}, {Ann, Bob, Clio}.

The set {Ann, Bob, Clio, Dai} has 16 subsets:

Ø, {Ann}, {Bob}, {Clio}, {Dai}, {Ann, Bob}, {Ann, Clio}, {Ann, Dai}, {Bob, Clio}, {Bob, Dai}, {Clio, Dai}, {Ann ,Bob , Clio}, {Ann, Bob, Dai}, {Ann, Clio, Dai}, {Bob ,Clio, Dai}, {Ann, Bob, Clio, Dai}.

In general, any set with n members has 2^n subsets.

To see why this should be so, imagine that you place the n members of some set A in a row, and that you then form a subset by going though these n members in turn deciding whether or not to include each in the subset. So you have two choices for the first member—in or out. And for each of these you have two choices for the second member—in or out. And for each of these four pairs of initial choices you have two choices for the third member…

So there are 2^n ways of forming a subset B. For each of the n members of the original set, you have a two-way yes–no option of whether to include it in your subset. (See Box 2.)

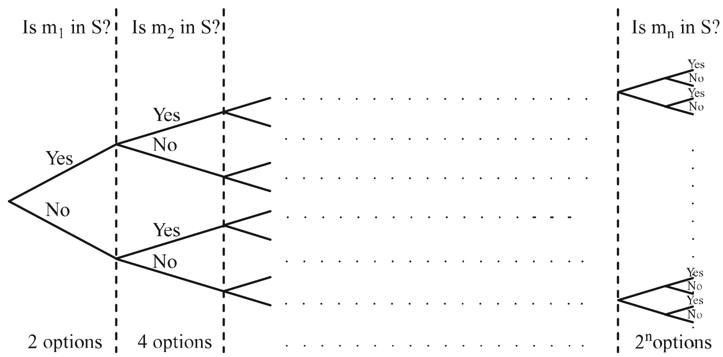
The set of all subsets of a set is called its *power* set. So the power set of a set with n members always has 2^n members.

So, as above, the power set of {Ann, Bob, Clio} is the 2^3-membered set {Ø, {Ann}, {Bob}, {Clio}, {Ann, Bob}, {Ann, Clio}, {Bob, Clio}, {Ann, Bob, Clio}}.

(Note how none of this would come out so nicely if we didn't count the empty set Ø as a subset of every set.)

Box 2 The Size of Power Sets

Imagine that $m_1, m_2, \ldots, m_n$ are the n members of our original set A, and that we want to form a subset S of this set. We then have n successive yes–no choices of whether to include these members in S, giving us altogether 2^n ways of forming S.



1.7 The Axiom of Comprehension

The 'axiom of extensionality' told us when two sets are the same—they have just the same members.

But how many sets are there in the first place? So far we have been assuming that, for any condition, there will be a set of things satisfying that condition.

The assumption that there exists a set for every condition can be made explicit as the *axiom of comprehension*:

For any condition C, there exists a set A such that (for any x)($x \in A$ iff x satisfies C).

You might be wondering why I am being so pedantic as to make this assumption explicit. Is it not obvious that there is a set of things satisfying any given condition? For example, if the condition is *being red*, then we have the set {x: x is red}; if the condition is *being a European country*, then we have the set {x: x is a European country}; if the condition is *being Margaret Thatcher or Tony Blair,* then we have the set {Margaret Thatcher, Tony Blair}; and so on. What could be more obvious?

However, far from being obvious, the axiom of comprehension cannot possibly be true. The idea that there is a set for every condition quickly leads to contradiction.

1.8 Russell's Set

Given that sets can themselves be members of sets, there is nothing to stop some sets being members of themselves. The set of *all sets with more than one member* is a member of itself, for instance—for this set will certainly have more than one member, and so it will be a member of itself.

The set of *all things which are not buses, say*, will similarly be a member of itself—since it is a set and therefore not a bus.

Many other sets, of course, will *not* be members of themselves. For example, the set of *all sets with only one member* will not be a member of itself—for this set will have many members and so not belong to itself. Or again, the *set of all buses* will not be a member of itself—for this will be a set and not a bus and so again not belong to itself.

Now consider the condition: *is not a member of itself*.

According to the axiom of comprehension, there must be a set corresponding to this condition, namely, R = {x: x not-∈ x}. R will contain precisely those things that are not members of themselves.

However, as Bertrand Russell first showed in 1901, the assumption that R exists generates an inconsistency. For we can prove both that R is a member of itself and that it is not.

1.9 Russell's Paradox

First let us prove that R is a member of itself.

(a) *Assume* R is *not* a member of itself.
(b) But then, since R contains all sets that are *not* members of themselves, it *is* a member of itself.
(c) So we have contradicted our assumption (a).
(d) So 'by reductio ad aburdum' we can conclude that (a) is false and R is a member of itself.

(A proof 'by reductio ad absurdum' is where you conclude that some temporary assumption made for the sake of the argument––here (a)–must be false since its truth would imply a contradiction. 'Reductio ad absurdum' is simply Latin for 'reduction to absurdity'.)

Now we can similarly prove that R is not a member of itself.

(a´) *Assume* R *is* a member of itself.
(b´) But then, since R contains only sets that are not members of themselves, it is *not* a member of itself.
(c´) So we have contradicted our assumption (a´).
(d´) So 'by reductio' we can conclude that (a´) is false and R is *not* a member of itself.

We have now proved both (d) that R is a member of itself and (d´) that R is not a member of itself. Something has gone badly wrong. This is Russell's paradox.

(Just to keep things straight, don't confuse the *final* contradiction *between* the conclusions (d) and (d´) with the *earlier* contradictions encountered *in the course of proving* (d) and (d´). The latter were merely consequences of the temporary assumptions (a) and (a´) respectively, and were used to conclude that (a) and (a´) must be false—since they led to contradiction. But the contradiction between (d) and (d´) isn't a result of some temporary assumption made for the sake of the argument. Rather it is forced on us by the existence of R, which in turn follows from the axiom of comprehension.)

1.10 Barbers and Sets

It will be helpful to compare Russell's paradox with the 'paradox of the barber'.

You tell me that there is a barber who shaves all and only those who do not shave themselves. I wonder whether he shaves himself. And so I reason:

(a) *Assume* he does *not* shave himself.
(b) But then he does shave himself (he shaves all those who do not shave themselves…).
(c) So we have contradicted our assumption (a).
(d) So 'by reductio' we conclude that he *does* shave himself.

And:

(a´) *Assume* he *does* shave himself.
(b´) But then he does not shave himself (he shaves only those who do not shave themselves…).
(c´) So we have contradicted our assumption (a´).
(d´) So 'by reductio' we conclude that he does *not* shave himself.

Your claim about the barber has led to a contradiction. But in this case it is clear enough how to react. The contradiction shows that there can be no such barber. You are full of nonsense. Your claim has been reduced to absurdity. Despite what you say, there can't be a barber who shaves all and only those who do not shave themselves.

Now, at first pass, Russell's paradox calls for the same response. There can't be a set of all things which are not members of themselves, for the assumption that such a set exists leads to a contradiction. But the trouble in this case is that we can't just leave it at that. For the assumption that there is a set of all things which are not members of themselves isn't just some spurious claim made in idle conversation, like your story about the barber. It is an inescapable consequence of what looked like an obvious assumption about sets, namely, the assumption that there is a set corresponding to every condition. If we are to reject the set of all things which are not members of themselves, we have no choice but to give up this axiom of comprehension.

Russell's paradox arises because sets are things and so the axiom of comprehension—there is a set corresponding to every condition on things—also applies to conditions on sets. But the set we get from a condition on sets will depend on what sets are available as candidate members to start with—which is precisely what the axiom of comprehensions was supposed to tell us. What we have seen is that this implicit circularity is not only worrisome but *vicious* in the sense that it generates contradictions.

1.11 Alternatives to Naive Set Theory

It is common to refer to the axioms of extensionality and comprehension as together comprising 'naive set theory'. Certainly these two assumptions seem to capture the intuitive notion of a set. Sets are defined by their members (extensionality) and there is a set for any characterizable plurality of things (comprehension).

But Russell's paradox shows that naive set theory is too naive. In particular, it shows that naive set theory contradicts itself. Some philosophers take this to be further evidence against the reality of sets. But most mathematicians and logicians respond by seeking to replace the intuitive notion of a set by a more sophisticated understanding which is free of inconsistency. This improved understanding must somehow avoid positing a set of all things that are not members of themselves, otherwise inconsistency will inevitably return. So modern set theories all modify the axiom of comprehension in one way or another so as to limit the range of admissible sets. We needn't go into details. From now on I shall simply assume that talk of sets has somehow been made consistent.

Box 3 Russell's Bombshell

In 1902, just as he was putting the finishing touches to the second volume of his *Basic Laws of Arithmetic*, the great German logician Gottlob Frege received a letter from Bertrand Russell about the set of all things that are not members of themselves. In an Appendix to the volume Frege said 'A scientist can hardly meet with anything more undesirable than to have the foundations give way just as the work is finished. I was put in this position by a letter from Mr Bertrand Russell when the work was nearly through the press.' In fact Frege himself never found a satisfactory way of dealing with Russell's paradox. But subsequent mathematicians and logicians, including Russell himself, have developed a number of different ways of avoiding it.

FURTHER READING

Eric Steinhart's *More Precisely: The Math You Need To Do Philosophy* (Broadview Press 2009) is a useful introductory complement to the present book. The first two chapters deal with basic set theory in rather more detail than I have.

Michael Potter's *Set Theory and its Philosophy* (Oxford University Press 2004) is an advanced philosophical introduction to the material covered in the first three chapters of this book.

Mary Tiles' *Philosophy of Set Theory; An Historical Introduction to Cantor's Paradise* (Dover Books 2004) covers much of the same ground.

EXERCISES

1. What is the union of the following pairs of sets?

   (a) {Abe, Bertha}, {Bertha, Carl}
   (b) {2, 5, 7, 11, 13}, {1, 5, 11, 13}
   (c) {x: x is a child aged 7–12}, {x: x is a child aged 10–15}
   (d) {France, Germany, Italy}, {Germany, Italy}
   (e) {France, Germany, Italy}, {India, China}
   (f) {x: lives in Germany}, {x: x lives in Europe}
   (g) {x: x lives in China}, {x: lives in Europe}
   (h) {x: x weighs more than 10 kilos}, {x: x weighs more than 7 kilos}

2. What is the intersection of each of the above pairs of sets?

3. List all the subsets of the following sets.

   (a) {Abe, Bertha}
   (b) {7, 8, 9}

4. Give the power sets of the following sets.

   (a) {1, 7}
   (b) {London, Manchester, Birmingham}

5. Consider the set {1, 2, 3, 7, 8}.

Which of the following items are (a) members, (b) subsets, or (c) neither?

2, {7, 8}, {2, 3}, {}, 3, {1, 2, 3, {7,8}}

6. Consider the set {1, 2, 3, {7, 8}, {2, 3}}.

Which of the following items are (a) members, (b) subsets, or (c) neither?

2, {7, 8}, {2, 3}, {}, 3, {1, 2, 3, {7, 8}}

7. (A): 'This sentence is false.'

Show carefully that this statement leads to a contradiction. (Hint: first assume that (A) is true, then assume that it is not true.)