Phil 5506: Metalogic
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BASICS OF PROPOSITIONAL META/LOGIC

1. Semantic Vocabulary

An **interpretation** of the language P assigns 'true' or 'false' (but not both) to each atomic wff of P. Since the usual meaning of the connectives is assumed, an interpretation thereby determines the truth-values for every wff of P.

An interpretation I is a **model** of a (non-empty) set of wffs Γ of the language P iff every member of Γ is true on I.

A wff α of P is **entailed by** or is a **semantic consequence of** a (possibly empty) set Γ iff there is no model for Γ where α is false. This is written as '$\Gamma \vDash_P \alpha$', although I drop the subscript on the double turnstile (so long as it is clear that the language in question is P).

α is **logically valid** in P or a **tautology** of P iff α is true on every interpretation, written as '$\vDash \alpha$'.

A set of wffs Γ of P is **satisfiable** or **m-consistent** ("model-theoretically consistent") iff there is a model for Γ. Otherwise, it is **unsatisfiable** or **m-inconsistent**.

2. Syntactic Vocabulary

A **derivation**[1] in PS of the wff α from the (possibly empty) set of wffs Γ is a finite, nonempty sequence of wffs $\varphi_0, \varphi_1, \ldots, \varphi_n$, ending in α, such that for every wff φ_i in the sequence, either:

i. φ_i is an axiom of PS, or
ii. φ_i is a member of Γ, or
iii. φ_i results from a single application of *modus ponens* to a pair of earlier wffs in the sequence (or: φ_i is an "immediate consequence" of earlier wffs).

α is **derivable** or is a **syntactic consequence** of Γ in PS, iff there is a derivation in PS of α from Γ. This is written as '$\Gamma \vdash_{PS} \alpha$', although I drop the subscript on the single turnstile (so long as it is clear that the formal system in question is PS).

α is a **theorem** iff α is derivable from the empty set in PS, written as '$\vdash \alpha$'. (Note that every axiom of PS is also a theorem.)

Γ is **p-consistent** ("proof-theoretically consistent") in PS iff there is no α such that both $\Gamma \vdash \alpha$ and $\Gamma \vdash {\sim}\alpha$. Otherwise, it is **p-inconsistent**.

3. Main Metalogical Results for PS

PS is **sound**: If $\Gamma \vdash \alpha$, then $\Gamma \vDash \alpha$.
PS is **complete**: If $\Gamma \vDash \alpha$, then $\Gamma \vdash \alpha$.

[1] Note that Hunter reserves the term 'proof' for derivations in which all wffs are theorems.