BASICS OF QUANTIFICATIONAL META/LOGIC

0. Unfamiliar features of Q

Wff in Q

Hunter's Q is unlike the quantificational logic you see in undergraduate logic texts. For instance, you get a new wff by adding "$\wedge$v" or "$\vee$v " to *any* pre-existing wff A. That's so, even if the variable v doesn't occur in A, or even if a quantifier binding v *already* occurs in A. Thus, the following are some wff of Q which you might find surprising.

$\wedge x'$ F$'x''$	$\wedge x' \wedge x' \wedge x'$ F$'x'$
$\wedge x'$ F$'a'$	$\vee x' \wedge x'$ F$'x'$
$\wedge x'$ p'	$\sim\vee x' \wedge x'$ F$'x'$

This is, in fact, entirely standard in mathematical logic; it is seen in *Principia Mathematica* for example. But it implies other things which might be unfamiliar, e.g., when we get to defining truth. In logic textbooks, truth/falsity apply to *sentences* (a.k.a., closed wff…having no free variables).[1] But in Q, some open wff are "true" and some are "false" too. Consequently, there are "logical validities" in Q with free variables; such wff are also among the "axioms" and "theorems" in the formal system QS.

I'm not sure what the advantages of this are, but never mind. Note, however, that the metatheorems end up stating something *stronger* than they would otherwise. E.g., completeness for QS implies that, since you can derive wff with free variables in QS as "theorems," these are "logically valid" in the present sense.

UG and UI in QS

Since open wff can be "true" in Q, universal generalization (UG) must be restricted in QS. We do not want [QS5] (= the schema for UG) to license the following *False Axiom:*

(FA) F$'x' \supset \wedge x'$ F$'x'$

Thus we stipulate that axioms from [QS5] must have no free variable in the antecedent. Now it is typical in natural deduction systems for UG to have restrictions to avoid overgeneralization. But the restriction in QS owes to the *open wff* among the axioms and theorems.

Another unfamiliar feature of QS might be that universal *instantiation* (UI) has restrictions. We shouldn't be able to generate from [QS4] (= the schema for UI) the following *False Axiom 2:*

(FA2) $\wedge x' \sim \wedge x''$ F$'x'x'' \supset \sim\wedge x''$ F$'x''x''$

This is false on some interpretations. E.g., if 'F$'$**' is identity, then (FA2) says that if everything is non-identical to something, then not everything is self-identical.

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[1] Take heed that a closed wff in Hunter is not necessarily the "closure" of a wff, as defined on p.139. Ugh.

Terms that are "free for" a variable.
Thus, we get a restriction on [QS4]: When instantiating a quantified variable v, use a term that is "free for" v. Basically, when removing the v-quantifier but before replacing v, you first notice that v is free…and you must replace it with a term t that remains "free" after the replacement. This means that any free-variable component of t should not become bound after the replacement. (I will elaborate in class.)

1. Semantic Vocabulary for Q

In Q, we cannot just assign truth-values to all non-truth-functional wff, since it will suggest possibilities that are not possible. Thus, '$\bigwedge x$ F$'x'$' and 'F$'a'$' are both non-truth functional, but you should not be able to assign 'true' to the former and 'false' to the latter. Thus, we adopt a more nuanced way to interpret a language with subject-predicate structure and quantifiers:

An **interpretation** *I* of the language Q specifies a non-empty set D, the **domain**, and assigns:

1. 'true' or 'false' (but not both) to each propositional symbol of Q.
2. some member of D to each constant of Q.
3. some total function on D to each functor of Q.[2]
4. a set of *n*-tuples of members in D to each *n*-place predicate of Q (known as the **extension** of the predicate).[3]

The truth-functional connectives have their usual meaning.

But, beyond the propositional symbols, we need further measures to determine which wff are true on *I*. The standard way to do this (following Tarski) is to determine which wff are "satisfied" on *I*, and then define truth/falsity with respect to that.

Yet first, we must define a denotation-function for the terms of Q on *I*, relative to a denumerable ordered set of elements from D. Let *s* be such a sequence $\langle e_1, e_2, e_3\ldots\rangle$.[4] Also, if x', x'', x'''… are the variables of Q, assume they are ordered by how many primes follow '*x*'. Given a term t of Q, the denotation-function t**s* yields an output as per the following rules:

- If t is the *k*th variable in the enumeration, then t**s* = the *k*th element in *s*.
- If t is a constant, then t**s* = the element of D assigned to t by *I*.
- If t is a functor, then t**s* = the function on D assigned to t by *I*.

[2] Partial functions on D can still be expressed by means of predicates (e.g., "*x* is the square root of *y*"). But since every term must refer to an object in classical logic, partial functions on D cannot be expressed by functors.
[3] Actually, if 'F' is a one-place predicate, then its extension consists in members of D, rather than *n*-tuples of such. Accordingly, in clause 2 of the definition of 'satisfaction', a wff 'Ft' featuring a one-place predicate 'F' is satisfied iff t**s* is a member of the extension for F. But following Hunter, I gloss these subtleties above.
[4] Annoyingly, Hunter refers to the elements of the sequence as the "terms" of the sequence. This makes 'term' equivocal between a linguistic item of Q and the denotation of such items. I reserve 'term' for the linguistic items.

We can give an inductive definition that a sequence *s* **satisfies** a wff A on *I* iff:

<u>Basis clauses</u>

1. A is a propositional symbol and *I* assigns 'true' to A; or
2. A is an atomic wff $Ft_1 \ldots t_n$, where F is an *n*-place predicate and $t_1 \ldots t_n$ are terms, and $\langle t_1{*}s \ldots t_n{*}s \rangle$ is a member of the extension assigned to the predicate by *I*; or

<u>Inductive clauses</u>

3. A is a wff ~B and *s* does not satisfy B on *I*; or
4. A is a wff B ⊃ C and either *s* does not satisfy B or *S* satisfies C on *I*; or
5. A is a wff ⋀v B and every sequence satisfies B on *I*; or
6. A is a wff ⋁v B and some sequence satisfies B on *I*.

The definition secures that, given an interpretation *I*, a wff of Q is either satisfied by *every* sequence, or satisfied by *none*. Accordingly, truth and falsity are defined thus:

- A wff of Q is **true** on *I* iff it is satisfied on *I* by every sequence.
- A wff of Q is **false** on *I* iff it is satisfied on *I* by no sequence.

NOTE: Some open wff are *neither true nor false*, e.g. F′*x*′. Even so, some open wff are true, e.g., F′*x*′ ⊃ F′*x*′ and ⋀*x*″ F′*x*″ ⊃ F′*x*′ (also, the negations of these are false).

With the definitions of true/false, we can then define other semantic notions for Q as one might expect:

An interpretation *I* is a **model** of a (non-empty) set of wffs Γ of the language Q iff every member of Γ is true for *I*.

A wff α of Q is **entailed by** or is a **semantic consequence of** a (possibly empty) set Γ iff there is no model for Γ where α is false. This is written as '$\Gamma \vDash_Q \alpha$', although I drop the subscript on the double turnstile (so long as it is clear that the language in question is Q).

α is **logically valid** in Q iff α is true on every interpretation, written as '$\vDash \alpha$'.

A set of wffs Γ of Q is **satisfiable** iff there is a model for Γ. Otherwise, it is **unsatisfiable.**

2. Syntactic Vocabulary for QS

The syntactic vocabulary for QS is entirely parallel to that for PS.

A **derivation** in QS of the wff α from the (possibly empty) set of wff Γ is a finite, nonempty sequence of wff $\varphi_0, \varphi_1, \ldots, \varphi_n$, ending in α, such that for every wff φ_i in the sequence, either:

i. φ_i is an axiom of QS, or

ii. φ_i is a member of Γ, or

iii. φ_i results from a single application of *modus ponens* to a pair of earlier wff in the sequence (or: φ_i is an "immediate consequence" of earlier wff).

α is **derivable** or is a **syntactic consequence** of Γ in QS, iff there is a derivation in QS of α from Γ. This is written as '$\Gamma \vdash_{QS} \alpha$', although I drop the subscript on the single turnstile (so long as it is clear that the formal system in question is QS).

α is a **theorem** iff α is derivable from the empty set in QS, written as '$\vdash \alpha$'. (Note that every axiom of QS is also a theorem.)

Γ is **p-consistent** ("proof-theoretically consistent") in QS iff there is no α such that both $\Gamma \vdash \alpha$ and $\Gamma \vdash {\sim}\alpha$. Otherwise, it is **p-inconsistent**.