REPRESENTATION IN H

Hunter's System H is (i) respectable, (ii) has a decidable set of wff, and (iii) a decidable set of proofs. It thus follows that *H is not negation-complete* (from Generalized Gödel Theorem), and that *H is undecidable* (from "Generalized Undecidability" Theorem). And since H is a finite extension of $QS^{=}$, the undecidability of H means that *$QS^{=}$ is undecidable* (Church's Theorem).

We consider (ii) and (iii) in a later handout. Here, we will establish (i). Recall that:
Definition: A formal system of arithmetic is **respectable** iff:

(a) The system is consistent.
(b) Every decidable set of natural numbers is represented in it.
(c) An open wff is a theorem iff some closure of it is.

Actually, we simply assume that (a) is true of H. Also, (c) is true of H since H is built to be a first-order theory in Hunter's sense. Thus, we will be concerned below only to show that (b) is true of H. [The Representation Theorem for H]

Preliminary: Recursion Theory

The premises behind the Representation Theorem for H will make reference to recursive sets and recursive functions…

Definitions. A set A is a **recursive set** iff its characteristic function f_A is a recursive function. In what follows, assume f_A is the **characteristic function** for a set A iff, for each $n \in \mathbb{N}$,

$$f_A(n) = \begin{cases} 0 & \text{if } n \in A \\ 1 & \text{otherwise} \end{cases}$$

Definition. A function f is a **recursive function** iff f is one of the base functions, or is obtainable from these by finitely many applications of composition and/or the μ-operation.[1]

Base: $x+1$ [successor], $x + y$ [addition], $x \cdot y$ [multiplication], x^y [exponentiation], $x \dot{-} y$ [arithmetic difference, i.e., $x - y$ if $x > y$; 0 otherwise].

Composition: h is obtained from functions f and g by composition iff:

$$h(\ldots, x, \ldots, y, \ldots) = f(\ldots, x, \ldots, g(\ldots, y, \ldots), \ldots)$$

-where any variable to the right of '=' also appears on the left.
Patently, if f and g are computable, then so is h.

μ-operation: Roughly, μ is a minimization operator that operates on a certain type of function f. Specifically, f must be *computable* and a *total* n+1-ary function on $\mathbb{N}$ where, for *any* n-tuple $\langle x_1, \ldots, x_n \rangle$, there is a y such that $f(x_1, \ldots, x_n, y) = 0$. Given such an f, an n-ary function g is obtained by the μ-operation iff:

[1] Usually, recursive functions also include those obtained by the operation of "primitive recursion." But the copious axioms of System H render such an operation unnecessary. (This, in turn, allows us to skip the notorious "Beta function lemma" in the proof of the Expressibility Lemma, thank god.)

$$g(x_1, \ldots, x_n) = \mu y\{f(x_1, \ldots, x_n, y) = 0\}$$

-where the right-hand expression means "the least y such that $f(x_1, \ldots, x_n, y) = 0$."
It is clear that any such g is computable.

Examples of recursive functions:
$f(x, y) = x\cdot(x \dot{-} y)$ $f_2(x, y, z) = (x\cdot(x \dot{-} y))^z$ $g(x) = \mu y\{x\cdot(x \dot{-} y) = 0\}$

But not:
$f_3(x) = \mu y\{x + y = 0\}$ [b/c no such y when $x > 0$]

56.20 Representation Theorem for H

56.20: Any decidable set of natural numbers is represented in H. [Representation Theorem]

Premises:
CT: Any decidable set is a recursive set. [Church's Thesis, in one of its formulations]
56.18: Any recursive function is represented in H.

Definition. A set X of natural numbers is **represented** in a formal system S iff there is a formula A, with just one free variable v, such that for each natural number n:

$\vdash_S A\underline{n}/v$ iff $n \in X$

Definition. An n-ary function f is **represented** in a formal system S iff there is a formula $A(v_1 \ldots v_{n+1})$ with n+1 free variables such that, for each n+1-tuple of natural numbers $\langle k_1 \ldots k_{n+1}\rangle$,

$\vdash_S A(\underline{k_1}, \ldots, \underline{k_{n+1}})$ iff $f(k_1 \ldots k_n) = k_{n+1}$.

The Basic Argument:
Given CT, the representation of a decidable set in H is really just a special case of the representation of a recursive function. Suppose that S is decidable. Then given CT, its characteristic function f_s is a recursive function. So by 56.18, f_s is represented in H by some formula $A(x, y)$ such that $\vdash_H A(\underline{n}, \underline{0})$ iff $f_s(n) = 0$. Since $\underline{0}$ is a name, this means there is a one-place formula $A(x, \underline{0})$ such that $\vdash_H A(\underline{n}, \underline{0})$ iff $n \in S$. That suffices for S to be represented in H.

56.17 The Correspondence or Expressibility Lemma for H

56.17: Any recursive function is strongly represented in H. [Expressibility Lemma or EL][2]

Definition. An n-ary function f is **strongly represented** or **(numeralwise) expressible** in a formal system S iff there is a formula $A(v_1, \ldots, v_{n+1})$ with n+1 free variables such that, for each n+1-tuple of natural numbers $\langle k_1, \ldots, k_{n+1}\rangle$, two conditions are met:

(i) $\vdash_S A(\underline{k_1}, \ldots, \underline{k_{n+1}})$ if $f(k_1, \ldots, k_n) = k_{n+1}$, and
(ii) $\vdash_S \sim A(\underline{k_1}, \ldots, \underline{k_{n+1}})$ if $f(k_1, \ldots, k_n) \neq k_{n+1}$.

[2] Hunter shows the even stronger result that recursive functions are (what he calls) "definable." But the extra strength is unnecessary and we'll skip it. (Caution: Boolos et al. refer to strong representability as "definability," and Tarski's "definability" theorem uses the term in yet a third sense.)

-56.18 will follow immediately from EL, assuming H is consistent.

Inductive Argument for EL

Premises: Where m and n are any natural numbers…

Def: The numeral for any natural number n is $\underline{n}$
56.5: $\vdash_H \ulcorner S\underline{n} = \underline{n+1}\urcorner$[3]
56.2: If $m < n$, then $\vdash_H \ulcorner \underline{m} < \underline{n}\urcorner$.
56.3: If $m \neq n$, then $\vdash_H \ulcorner {\sim}\underline{m} = \underline{n}\urcorner$.
56.6: $\vdash_H \ulcorner \underline{m} + \underline{n} = \underline{m+n}\urcorner$.

Basis: Each base recursive function is strongly represented in H. (5 cases).

Case (1): '$Sx = y$' strongly represents the successor function. (Proof left as an exercise.)
Case (2): '$x + y = z$' strongly represents addition.

Condition (i): Suppose $m + n = k$. Then, $\underline{m+n} = \underline{k}$, by definition. And by 56.6, $\vdash_H \ulcorner \underline{m} + \underline{n} = \underline{m+n}\urcorner$. Therefore, $\vdash_H \ulcorner \underline{m} + \underline{n} = \underline{k}\urcorner$.

Condition (ii): Suppose $m + n \neq k$. Then by 56.3, $\vdash_H \ulcorner {\sim}\underline{m+n} = \underline{k}\urcorner$. Since by 56.6, $\vdash_H \ulcorner \underline{m} + \underline{n} = \underline{m+n}\urcorner$, axiom 3 assures us that $\vdash_H \ulcorner {\sim}\underline{m} + \underline{n} = \underline{k}\urcorner$.

Case (3): '$x \cdot y = z$' strongly represents multiplication.
Case (4): '$Pxy = z$' strongly represents exponentiation.
Case (5): '$(y < x \wedge x = y + z) \vee ({\sim}y < x \wedge z = 0)$' strongly represents arithmetic difference.

Inductive Step: We want to show that if function h is obtained by $n > 0$ applications of combination or the μ-operation, then h is strongly represented in H.

Preliminary Lemma [PL]: Suppose $A(x, y)$ is a wff of H such that

$$\vdash_H \bigwedge x \bigwedge y \bigwedge z\, (A(x, y) \supset (A(x, z) \supset y = z))$$

Then, a unary function f is strongly represented in H by $A(x, y)$ if $\vdash_H \ulcorner A(\underline{m}, \underline{f(m)})\urcorner$.

Pf. of PL:

Condition (i). Suppose $f(m) = n$ so that $\underline{f(m)}$ is $\underline{n}$. Then, if $\vdash_H \ulcorner A(\underline{m}, \underline{f(m)})\urcorner$, $\vdash_H \ulcorner A(\underline{m}, \underline{n})\urcorner$.

Condition (ii). Suppose $f(m) \neq n$. Then by 56. 3, $\vdash_H \ulcorner {\sim}\underline{f(m)} = \underline{n})\urcorner$. Also, since we assume the antecedent of PL, we know $\vdash_H \ulcorner A(\underline{m}, \underline{f(m)}) \supset (A(\underline{m}, \underline{n}) \supset \underline{f(m)} = \underline{n})\urcorner$. So by the logical axioms, $\vdash_H \ulcorner A(\underline{m}, \underline{f(m)}) \supset ({\sim}\underline{f(m)} = \underline{n} \supset {\sim}A(\underline{m}, \underline{n}))\urcorner$. Thus, if $\vdash_H \ulcorner A(\underline{m}, \underline{f(m)})\urcorner$, then by MP twice, $\vdash_H \ulcorner {\sim}A(\underline{m}, \underline{n})\urcorner$.

Combination: [We consider only functions of one argument, but the reasoning generalizes to all functions.] The inductive hypothesis is that f and g are recursive unary functions that strongly represented in H by the formulae $A(x, y)$ and $B(x, y)$, respectively. We aim to show that the function $h(x) = f(g(x))$ is represented in H, specifically, by the formula '$\bigvee w(B(x, w) \wedge A(w, y))$'.

[3] The corner-quoted expression denotes the result of replacing the numeral-variables inside the corner quotes with the relevant numeral(s). For instance, if n = 0+0, then $\ulcorner S\underline{n} = \underline{n+1}\urcorner$ is the expression 'S0 = 0″'.

Since A(x, y) and B(x, y) strongly represent functions, they satisfy the antecedent of PL. (Trust me on this.) From that, we can show $\bigvee w(\mathrm{B}(x, w) \wedge \mathrm{A}(w, y))$ also satisfies the antecedent of PL. (You'll need to trust me here too.) So as per the consequent of PL, it suffices to show $f(g(x))$ is strongly represented in H by proving that $\vdash_{H} \ulcorner \bigvee w(\mathrm{B}(\underline{m}, w) \wedge \mathrm{A}(w, \underline{f(g(m))})) \urcorner$.

By the strong representation of f and g, we know $\vdash_{H} \ulcorner \mathrm{B}(\underline{m}, \underline{g(m)}) \urcorner$, and that $\vdash_{H} \ulcorner \mathrm{A}(\underline{g(m)}, \underline{f(g(m))}) \urcorner$. So by $\wedge$-introduction, $\vdash_{H} \ulcorner \mathrm{B}(\underline{m}, \underline{g(m)}) \wedge \mathrm{A}(\underline{g(m)}, \underline{f(g(m))}) \urcorner$. And thus by existential generalization, $\vdash_{H} \ulcorner \bigvee w(\mathrm{B}(\underline{m}, w) \wedge \mathrm{A}(w, \underline{f(g(m))})) \urcorner$.

μ-Operation: [We consider only functions of two arguments, but…etc.] Assume f is a recursive binary function where for each m, there is an n such that $f(m, n) = 0$. And assume f is strongly represented in H by the formula 'A(x, y, z)'. We aim to show that $h(m) = \mu n\{f(m, n) = 0\}$ is strongly represented in H, specifically, by '$\mathrm{A}(x, y, 0) \wedge \bigwedge w\, (w < y \supset {\sim}\mathrm{A}(x, w, 0))$'.

Since 'A(x, y, z)' strongly represents a function, it satisfies the antecedent of PL. Therefore, so does '$\mathrm{A}(x, y, 0) \wedge \bigwedge w\, (w < y \supset {\sim}\mathrm{A}(x, w, 0))$'. (I invoke your trust here as well.) So as per the consequent of PL, it suffices to show $h(m)$ is strongly represented in H by proving that $\vdash_{H} \ulcorner \mathrm{A}(\underline{m}, \underline{h(m)}, 0) \wedge \bigwedge w(w < \underline{h(m)} \supset {\sim}\mathrm{A}(\underline{m}, w, 0)) \urcorner$.

First conjunct: By the definition of h, we know that $f(m, h(m)) = 0$, for every m. And so, by the representability of f, we know that $\vdash_{H} \ulcorner \mathrm{A}(\underline{m}, \underline{h(m)}, 0) \urcorner$.

Second conjunct: Suppose for conditional proof that $w < h(m)$, for an arbitrary w. Then, by the definition of h and f, $f(m, w) \neq 0$. So by the strong representation of f, $\vdash_{H} \ulcorner {\sim}\mathrm{A}(\underline{m}, \underline{w}, 0) \urcorner$. Thus, by logical axioms, $\vdash_{H} \ulcorner \underline{w} < \underline{h(m)} \supset {\sim}\mathrm{A}(\underline{m}, \underline{w}, 0) \urcorner$. Since $\underline{w}$ is arbitrary, universal generalization secures that $\vdash_{H} \ulcorner \bigwedge w(w < \underline{h(m)} \supset {\sim}\mathrm{A}(\underline{m}, w, 0)) \urcorner$.[4]

[4] Caveat: System H does not allow universal generalization in the form I just used. Instead, the maneuver above would require several more steps, but I'm just skipping all that.