COMPLETENESS AND ITS COMPANIONS

PS is **complete**: If $\Gamma \vDash_P$ A, then $\Gamma \vdash_{PS}$ A. I.e., If Γ entails A, then A is derivable from Γ in PS, (where A is a wff in P, and Γ is a [possibly empty] set of wff in P). Completeness is the converse of soundness. Subscripts on '⊨' and '⊢' omitted in what follows.

32.14: The Completeness Theorem for PS

Premises
32.13: If Γ is p-consistent in PS, then Γ has a model. [The Model Existence Lemma]
32.7: Γ ⊢ A iff Γ ∪ {~A} is not p-consistent in PS.
-Hereafter, I drop the 'in PS' and just speak of p-consistency.

The Basic Argument:

(1) Γ ⊨ A.	[Suppose for conditional proof]
(2) Γ ∪ {~A} has no model.	[From (1)]
(3) Γ ∪ {~A} is p-inconsistent.	[From (2) and 32.13]
(4) Γ ⊢ A.	[From (3) and 32.7]
(5) So, PS is complete.	[By conditional proof (1)-(4)]

Remark: This proof is **non-constructive**: For every entailment, it shows there must be a derivation. But it does not tell us how the derivation is built. (Contrast with the proof of the Deduction Theorem.)

The proof of 32.7 is fairly short (see n. 1); the majority of the completeness proof consists in the demonstrating of the Model Existence Lemma.

32.13 The Model Existence Lemma for PS

The Overall Strategy.
Start with Γ, a set of wff which is p-consistent…
1. Build from Γ a set Γ* of wff that is *maximally* p-consistent.
2. Show that the maximally p-consistent set Γ* has a model.

It follows that Γ must have a model as well.

Definition: Γ is *maximally p-consistent* iff Γ is p-consistent and, for any wff A ∉ Γ, Γ ∪ {A} is p-inconsistent.

To complete the Strategy. *We must show*:
32.12: Any p-consistent set Γ is a subset of a maximally p-consistent set Γ*. [Lindenbaum's Lemma]
MXL: If Γ* is maximally p-consistent, it has a model. [Model *Extraction* Lemma]

32.12 Lindenbaum's Lemma for PS

Preliminary: We will assign each symbol of P (and hence, each wff) a unique numeric code, according to the following table:

Symbol	*Numeral*	*Symbol*	*Numeral*
p	1	⊃	4
′	2	(	5
~	3	)	6

Each wff is coded by the concatenation of the numerals for each of its symbols. Thus, '~(*p*′ ⊃ *p*″)' uniquely receives the code '351241226'. Accordingly, we can speak of an "enumeration" of wffs, where the first wff is the one with the lowest numeral (viz., the atomic wff '*p*″'), the second wff is the one with the next lowest numeral, etc. This indicates an algorithm for identifying the numeric code for any wff. One can also find an algorithm for identifying the wff, if any, for a numeral.

*Building Γ**: For a given set Γ that is p-consistent, a set Γ* that is maximally p-consistent can be defined as the *union* of an infinite sequence of sets $\Gamma_0, \Gamma_1, \Gamma_2\ldots$, where this infinite sequence is defined inductively as follows:

1. Γ is the first set in the sequence; call it Γ_0.
2. If A_{n+1} is the *n*th sentence in our enumeration, then $\Gamma_{n+1} = \Gamma_n \cup \{A_{n+1}\}$, provided that $\Gamma_n \cup \{A_{n+1}\}$ is p-consistent; otherwise $\Gamma_{n+1} = \Gamma_n$.

Basically, Γ* is going to be formed by taking Γ and successively expanding it in a way that preserves p-consistency, where every wff is considered for membership in the expansion, in order of the enumeration.

Proof that Γ* is p-consistent
If Γ* is p-inconsistent, then since the p-consistency of Γ_0 is given, it must be that a wff was added that resulted in a p-inconsistent set. This means that one of $\Gamma_1, \Gamma_2, \Gamma_3\ldots$ is p-inconsistent. But all of those sets are p-consistent, by clause 2 of the definition of Γ*.

Sub-Lemma: For any *n*, $\Gamma_n \subseteq \Gamma^*$ and if $\Gamma^* \cup \{A_n\}$ is p-consistent, so is $\Gamma_n \cup \{A_n\}$.
Proof: The 1st conjunct holds by def. of the series $\Gamma_0, \Gamma_1, \Gamma_2\ldots$. Re: the 2nd conjunct, since $\Gamma_n \subseteq \Gamma^*$ by def., then $\Gamma_n \cup \{A_n\} \subseteq \Gamma^* \cup \{A_n\}$. And if no contradiction is derivable from a set, then this holds of its subsets. Thus, if $\Gamma^* \cup \{A_n\}$ is p-consistent, so is $\Gamma_n \cup \{A_n\}$.

Proof that Γ* is maximally p-consistent
(1) Some wff $A_k \notin \Gamma^*$ is such that $\Gamma^* \cup \{A_k\}$ is p-consistent. [Assume for *reductio*]
(2) $\Gamma_k \subseteq \Gamma^*$ and $\Gamma_k \cup \{A_k\}$ is p-consistent. [From (1) and Sub-Lemma]
(3) $\Gamma_k \cup \{A_k\} = \Gamma_{k+1}$. [From (2) and by def. of the series]
(4) $A_k \in \Gamma_{k+1} \subseteq \Gamma^*$. [From (3) and the def. of Γ*]
(5) No wff $A_k \notin \Gamma^*$ is such that $\Gamma^* \cup \{A_k\}$ is p-consistent. [By *reductio*; cf. (1) and (4)]

MXL: The Model Extraction Lemma for PS

Once we've shown this, we're done proving Completeness. Do you see why?

Let *I* be the interpretation which assigns "true" to every atomic wff of P that is a member of Γ*, and assigns "false" to every other atom. We will show that for any wff A, A ∈Γ* iff A is true on *I*. This will establish that Γ∗ has a model.

Premises
32.9: For any maximal p-consistent set Γ* and wff A, A ∈ Γ* or ~A ∈ Γ* (but not both). ["Exhaustion" Lemma]
32.10: For any maximal p-consistent set Γ* and wff A, if Γ* ⊢ A then A ∈ Γ*. ["Explicitation" Lemma]

The Argument for MXL:
Basis: An atomic wff is a member of Γ* iff it is true on *I*. (By definition of *I*).

Inductive step: The inductive hypothesis (IH) is that any wff with <*k* connectives is a member of Γ* iff it is true on *I*. The aim is to show that a wff with *k* connectives A ∈ Γ* iff A is true on *I*. We consider two cases, corresponding to the two connectives that could be added as the k^{th} connective: (i) A has the form ~B, (ii) A has the form B ⊃ C.

Case (i):
[L to R] Suppose for conditional proof that ~B ∈ Γ*; we want to show that ~B is true on *I*. Now it must be that B ∉ Γ*, since Γ* is p-consistent. Yet B has <*k* connectives, and so by IH, B is true on *I* iff B ∈ Γ*. So B is not true; hence, ~B is true.

[R to L] Suppose for conditional proof that ~B is true on *I*; we want to show that ~B ∈ Γ*. Now it must be that B is false on *I*, and since B has <*k* connectives, IH assures us that B is true on *I* iff B ∈ Γ*. So B ∉ Γ*. And by Exhaustion, ~B ∈ Γ*.

Case (ii):
[L to R] Suppose for conditional proof that B ⊃ C ∈ Γ*; we want to show that B ⊃ C is true on *I*. Assume otherwise for *reductio*. Then, B is true and C is false. Since B and C each have <*k* connectives, IH applies to each. So B ∈ Γ* and C ∉ Γ*. By Exhaustion this means ~C ∈ Γ*. Hence, Γ* ⊢ B and Γ* ⊢ ~C. Note also that ⊢ (B ⊃ (~C ⊃ ~(B ⊃ C)). So by MP twice, Γ* ⊢ ~(B ⊃ C). Yet by the p-consistency of Γ*, this means B ⊃ C ∉ Γ*, contra our initial supposition. Hence, by *reductio*, B ⊃ C is true on *I*.

[R to L] Suppose for conditional proof that B ⊃ C is true on *I*; we want to show that B ⊃ C ∈ Γ*. Now it must be that either B is false or C is true, and since each has <*k* connectives, IH applies. So either B ∉ Γ* or C ∈ Γ*…

Assume the former disjunct. By Exhaustion, ~B ∈ Γ*, thus, Γ* ⊢ ~B. Consider also that ⊢ ~B ⊃ (B ⊃ C). So by MP, Γ* ⊢ B ⊃ C. Hence, by Explicitation, B ⊃ C ∈ Γ*.

Assume the latter disjunct. Then Γ* ⊢ C. By [PS1], C ⊃ (B ⊃ C) is an axiom. So by MP, Γ* ⊢ B ⊃ C. Thus, by Explicitation, B ⊃ C ∈ Γ*.

Proof of 32.9: "Exhaustion" Lemma for PS
Premises[1]
32.7: $\Gamma \cup \{\sim A\}$ is p-inconsistent iff $\Gamma \vdash A$.
32.8: $\Gamma \cup \{A\}$ is p-inconsistent iff $\Gamma \vdash \sim A$.

Suppose for *reductio* that neither A nor ~A are in Γ*. Then since Γ* is *maximally* p-consistent, neither can be added to Γ* without p-inconsistency. If A cannot be added without p-inconsistency, then by 32.8, $\Gamma^* \vdash \sim A$. And if ~A cannot be added without p-inconsistency, then by 32.7, $\Gamma^* \vdash A$. But those two derivational claims contradict the p-consistency of Γ*; so by *reductio*, one of A or ~A is in Γ*.

Proof of 32.10: "Explicitation" Lemma for PS
Assume $\Gamma^* \vdash A$ but suppose that $A \notin \Gamma^*$. Then by Exhaustion, $\sim A \in \Gamma^*$, hence, $\Gamma^* \vdash \sim A$. And so Γ* is p-inconsistent, contra assumption. Thus, if $\Gamma^* \vdash A$, then $A \in \Gamma^*$.

<u>32.18 The Finiteness Theorem for PS</u>

The soundness and completeness of PS imply that '$\vDash_P$' and '$\vdash_{PS}$' are interchangeable. Thus, since 23.7 is true (see the handout on Soundness), the parallel claim that follows is:

32.18: $\Gamma \vDash A$ iff there is a finite subset Δ of Γ such that $\Delta \vDash A$.

Remark: It doesn't matter if Γ is infinite: For *any* wff A that Γ entails, some finite set of wff entails A too. (Contrast with 23.7, a trivial result from the finitude of derivations.)

<u>32.20 The Compactness Theorem for P</u>

<u>Premises</u>:
32.13: If Γ is p-consistent, then Γ has a model. [The Model Existence Lemma]
32.19: If every finite subset of Γ is p-consistent, then Γ has a model.
∴ 32.20 If every finite subset of Γ has a model, then Γ has a model. [From 32.13 and 32.19] (Since the other direction of 32.20 is obvious, an 'iff' would be just as well.)

Proof of 32.19
Suppose that every finite subset of Γ is p-consistent, but that Γ is p-inconsistent. Then, for some wff A, A and ~A are both derivable from Γ. But derivations must be finite; hence, Γ has a finite subset Δ from which A and ~A are both derivable. But this is contra the initial supposition. So under that supposition, Γ must be p-consistent.

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[1] The proofs for both premises are parallel, and R to L directions are obvious. With 32.7, L to R is as follows: If $\Gamma \cup \{\sim A\}$ is p-inconsistent, then for some wff B, B and ~B are each derivable from $\Gamma \cup \{\sim A\}$. Thus, by our ol' pal the Deduction Theorem, $\Gamma \vdash \sim A \supset B$ and $\Gamma \vdash \sim A \supset \sim B$. And it is a theorem that $(\sim A \supset B) \supset ((\sim A \supset \sim B) \supset A)$; see Hunter p. 106-7 for details. So by MP twice, $\Gamma \vdash A$.