COMPLETENESS OF QS AND QS⁼

QS is **complete**: If $\Gamma \vDash_Q A$, then $\Gamma \vdash_{QS} A$. I.e., If Γ entails A, then A is derivable from Γ in QS, (where A is a wff in Q, and Γ is a [possibly empty] set of wff in Q). Completeness is the converse of soundness.

However, we shall prove completeness for *any* formal system with certain features, known as "first order theories."

Definition. A **first order theory K** is any formal system such that: [1]
- (i) K is defined on wff of Q or Q+, except that some or all of the propositional symbols, functors, and constants may be absent. Also, some—but not all—of the predicates may be absent.
- (ii) K has MP as its sole rule of inference. It also has wff generated from [QS1]-[QS7] as axioms, called the *logical* axioms. (N.B., [QS7] is revised to "If A is a *logical* axiom…") But we call the schemata for the logical axioms [K1]-[K7].
- (iii) K may also include any number of closed wff as *proper* axioms. K can be expanded by adding a sentence A of Q as a proper axiom, written 'K++{A}'.

Note: QS is a first-order theory by this definition.

32.14: The Completeness Theorem for K

Premises
45.6: If A is not a theorem of K, then K++{~A} is consistent.
MEL: If Γ is consistent in K, then Γ has a model. [Model Existence Lemma for K]

The Basic Argument:

(1) $K \vDash A$.	[Suppose for conditional proof]
(2) K++{~A} has no model.	[From (1)]
(3) K++{~A} is inconsistent.	[From (2) and MEL]
(4) $\vdash_K A$.	[From (3) and 45.6]
(5) So, K is complete.	[By conditional proof (1)-(4)]

Remark: This proof is **non-constructive**: For every entailment, it shows there must be a derivation. But it does not tell us how the derivation is built.

The proof of 45.9 is short; most of the completeness proof consists in showing MEL.

[1] Weirdly, Hunter omits what distinguishes a *first* order theory from any higher order theory, viz., that terms of first order theories denote only individuals, and not properties/relations of individuals (nor properties/relations of such properties/relations, etc.) Perhaps Hunter assumed this is first order aspect is essential to the languages Q and Q+…but since these are *formal* languages, and the aspect in question is *semantic*, he shouldn't view the first order aspect as essential to such languages.

32.13 The Model Existence Lemma for K

The Overall Strategy.
Start with a consistent first-order theory K…
(1) Build K*, a consistent extension of K that is *negation-complete* and *closed.*
(2) Show that K* has a model.
It follows that K must have a model as well.

Definition: A formal system S* is an **extension** of a formal system S iff every theorem of S is also a theorem of S*.
Definition: A formal system S is **negation complete** iff, for any *closed* wff A of S (a.k.a., sentence of S), either A is a theorem of S or ~A is a theorem of S.
Definition: Where S is a formal system with at least one closed term, S is **closed** iff for any wff A with *one* free v, if $\vdash_S$ At/v for *every* closed term t, then $\vdash_S$ $\bigwedge$vA.

To complete the Strategy. *We must show*:
32.12: For any consistent K, there is a consistent extension K* that is negation-complete and closed. [Lindenbaum's Lemma for K]
45.14: If a consistent system K is closed and negation-complete, it has a model. [Model *Extraction* Lemma for K]

32.12 Lindenbaum's Lemma for K

Preliminary:
We will assign each symbol of the language (and hence, each wff) a unique numeric code, according to the following table:

Symbol	*Numeral*	*Symbol*	*Numeral*
p	10	*	10000000
′	100	~	100000000
x	1000	⊃	1000000000
a	10000	$\bigwedge$	10000000000
f	100000	(	100000000000
F	1000000	)	1000000000000

Also, the denumerable constants of Q+ are coded in order of their recursive enumeration, starting with 10000000000000 for the first such constant, and increasing by orders of 10.

Each wff is coded by the concatenation of the numerals for each of its symbols. Accordingly, we can speak of an "enumeration" of wffs, where the first wff is the one with the lowest numeral, the second wff is the one with the next lowest numeral, etc. This indicates an algorithm for identifying the numeric code for any sentence. One can also find an algorithm for identifying the sentence, if any, for a numeral.

—Two-Phase Construction—
For any consistent K, we will first show how to build a consistent extension K* that is *negation-complete*. Also, given any consistent extension of K, we will show how to build a consistent extension K_∞ that is *closed*. It will then follow that for any consistent K, we may build a consistent extension K_∞* that is both negation-complete and closed.

Phase 1: *Building K.** First, let us isolate an enumeration of the closed wff (sentences) from the earlier enumeration; call it E. Then, given a consistent K, a consistent and negation-complete extension K* can be defined as the *union* of an infinite sequence of sets K_0, K_1, K_2…, where this infinite sequence is defined inductively as follows:

1. K is the first set in the sequence, K_0.
2. If A_{n+1} is the *n*th sentence in E, then $K_{n+1} = K_n$ with A_{n+1} "added" as an axiom [notation: K_n++A_{n+1}], provided that $\sim A_{n+1}$ is not a theorem of K_n. Otherwise, $K_{n+1} = K_n$.

Basically, K* is going to be formed by taking K and successively expanding its closed proper axioms in a way that preserves consistency, where every sentence is considered as a proper axiom in the expansion, in order of the enumeration.

Proof that K is consistent*
If K* is inconsistent, then since the consistency of K_0 is given, an axiom must have been added that resulted in an inconsistent system. This means that one of K_1, K_2, K_3… is inconsistent. But all of those are consistent, by clause 2 of the definition of K*.

Proof that K is negation-complete*
Take C_m, which is the *m*th sentence in the enumeration E. By definition of the series, $\sim C_m$ is a theorem of K_{m-1} (and hence of K*), or C_m is a theorem of K_m (and hence of K*). This holds of each positive integer *m*, hence for each sentence of K.

Phase 2: *Building K_∞*. Given a consistent K, let K_0 be the result of adding to K any missing constants from Q+, but no new proper axioms. K_0 is consistent.[2] Given our initial enumeration, we can isolate an enumeration of the wffs of K_0 with exactly one free variable. Let $\langle A_1, A_2, A_3\ldots\rangle$ be this enumeration. And let $\langle b_1, b_2, b_3\ldots\rangle$ be an enumeration of the constants of K_0; call the latter enumeration B. Then, define a denumerable series of sentences S_n as follows:

> <u>Basis</u>: $S_1 = A_1 b_{j1}/v_1 \supset \bigwedge v_1 A_1$, where b_{j1} is the first constant in B that does not occur in A_1.
> <u>Inductive Clause</u>: If $n > 1$, then $S_n = A_n b_{jn}/v_n \supset \bigwedge v_n A_n$, where b_{jn} is the first constant in B that does not occur in any of $S_1 \ldots S_{n-1}$.

[2] In adding new names, [K1]-[K7] still yield only logical validities. The soundness proof for QS did not depend on which names were part of the language. So K is also sound if its proper axioms are consistent.

Let K_1 be the result of adding S_1 as a proper axiom to K_0, and K_{n+1} the result of adding S_{n+1} as a proper axiom to K_n. And let K_∞ be the system that results from adding all such S_n to K_0 as proper axioms. (Note that K_∞ is clearly an extension of K_0.)

Proof that K_∞ is closed
Suppose for conditional proof that $A_k c/v_k$ is a theorem of K_∞, for *every* closed term c of K_∞. Then it is a theorem for the constant b_{jk} that occurs in $A_k b_{jk}/k_m \supset \bigwedge v_k A_k$…this being an axiom of K_∞ by definition. Via MP, then, it is also a theorem of K_∞ that $\bigwedge v_k A_k$.

Proof(?) that K_∞ is consistent.
Suppose for *reductio* that K_∞ is inconsistent. Then, since the consistency of K_0 is known, an axiom must have been added later that resulted in an inconsistent system. This means that one of K_1, K_2, K_3… is inconsistent. Suppose it is K_1. Then, the inconsistency resulted from adding $A_1 b_{j1}/v_1 \supset \bigwedge v_1 A_1$ to K_0. This means K_0 has as a theorem $\sim(A_1 b_{j1}/v_1 \supset \bigwedge v_1 A_1)$. And by the truth-functional axioms, this means K_0 also has as theorems A_1 and $\sim\bigwedge v_1 A_1$ (where v_1 is free in A_1). But by the soundness of K (cf. note 1), this means A_1 is logically valid, hence, that all sequences satisfy A_1 on all interpretations. And that suffices for the logical validity of $\bigwedge v_1 A_1$ as well. And so, by soundness again, $\sim\bigwedge v_1 A_1$ is not a theorem of K. Contradiction. So K_1 couldn't be the earliest inconsistent system in the series.

The same argument can be given for all other K_m where $m > 1$. So none of the K in the series are inconsistent. Hence, by *reductio*, K_∞ is consistent.

45.14: The Model Extraction Lemma for K

Let $K_\infty^* = T$, where T is a consistent, closed, and negation-complete extension of K, with denumerably many individual constants. We will construct an interpretation *I*, for which it can be shown that:

45.14: For any wff A of T, $\vdash_T A$ iff A is true on *I*.

That will suffice to establish that T has a model.

Extracting the Model. Let *I* be the following interpretation:

1. The domain D of *I* is the set of *closed terms* of T.
2. Each constant *c* denotes itself on *I*.
3. Where $t_1 \ldots t_n$ are closed terms, each closed functor $f_i t_1 \ldots t_n$ denotes itself on *I*. (More broadly, an *n*-place functor f_i expresses a function which takes an *n*-tuple of closed terms $\langle t_1 \ldots t_n \rangle$ as input, and outputs that very functor concatenated with those very terms, in the same order.)
4. Each predicate F_i has as its extension on *I* the set of *n*-tuples of closed terms $\langle t_1 \ldots t_n \rangle$ such that $\vdash_T F_i\, t_1 \ldots t_n$.
5. Each propositional symbol p_i is true on *I* iff $\vdash_T p_i$.

Premises
40.7: A is true on an arbitrary *I* iff any closure of A is true on *I*. ["SuperTruth" Lemma]
45.5 A is a theorem of a first order theory K iff any closure of A is a theorem of K. ["SuperTheorem" Lemma]

The Argument for 45.14:
We need only consider the closed wff of T, given 40.7 and 45.5 above.

Basis: A wff A with 0 connectives/quantifiers (a.k.a., "operators") is a theorem of T iff A is true on *I*. (Trivial in the case of propositional symbols. With other atomic wff, this is true by definition of the predicates; cf. clause 4 above).

Inductive step: The inductive hypothesis (IH) is that any wff with $<k$ operators is a theorem of T iff it is true on *I*. The aim is to show that a wff with k operators is a theorem of T iff A is true on *I*. We consider three cases, corresponding to the three operators that could be added as the k^{th} operator: (i) A has the form ~B, (ii) A has the form B ⊃ C, (ii) A has the form $\bigwedge$vB.

Case (i):
[L to R] Suppose for conditional proof that $\vdash_T$ ~B; we want to show that ~B is true on *I*. Now B must not be a theorem, since T is consistent. Yet B has $<k$ operators, and so by IH, B is true on *I* iff $\vdash_T$ B. So B is not true; thus, ~B is true.

[R to L] Suppose for conditional proof that ~B is true on *I*; we want to show that $\vdash_T$ ~B. Now B must be false on *I*, and since B has $<k$ operators, IH assures us that B is true on *I* iff $\vdash_T$ B. So B is not a theorem of T. By the negation-completeness of T, then, $\vdash_T$ ~B.

Case (ii):
[L to R] Suppose for conditional proof that $\vdash_T$ B ⊃ C; we want to show that B ⊃ C is true on *I*. Assume otherwise for *reductio*. Then, B is true and C is false. Since B and C each have $<k$ operators, IH applies to each. So $\vdash_T$ B and C is not a theorem of T. By the negation-completeness of T, this means $\vdash_T$ ~C. Note also that $\vdash_T$ (B ⊃ (~C ⊃ ~(B ⊃ C)). So by MP twice $\vdash_T$ ~(B ⊃ C). Yet by the consistency of Γ*, this means B ⊃ C is not a theorem of T, contra our initial supposition. Hence, by *reductio*, B ⊃ C is true on *I*.

[R to L] Suppose for conditional proof that B ⊃ C is true on *I*; we want to show that $\vdash_T$ B ⊃ C. Now it must be that either B is false or C is true, and since each has $<k$ operators, IH applies. So either B is not a theorem of T or $\vdash_T$ C …

Assume the former disjunct. Then, by the negation-completeness of T, $\vdash_T$ ~B. Consider also that $\vdash_T$ ~B ⊃ (B ⊃ C). So by MP, $\vdash_T$ B ⊃ C.

Assume the latter disjunct. By [K1], C ⊃ (B ⊃ C) is an axiom. So by MP, we know that $\vdash_T$ B ⊃ C.

Case (iii):
Either B is closed or B is has one free variable v…
[L to R, if B is closed] Suppose for conditional proof that $\vdash_T \bigwedge$vB; we want to show that $\bigwedge$vB is true on *I*. Now our supposition indicates (via [K4] and MP) that $\vdash_T$ B. And since B has <*k* operators, IH tells us that B is true on *I* iff $\vdash_T$ B. So B is true on *I*, and by SuperTruth, this means $\bigwedge$vB is true on *I*.

[R to L, if B is closed] Suppose for conditional proof that $\bigwedge$vB is true on *I*; we want to show that $\vdash_T \bigwedge$vB. Nour our supposition indicates by SuperTruth that B is true. And since B has <*k* operators, IH tells us that B is true on *I* iff $\vdash_T$ B. And so it follows that that $\vdash_T$ B. Hence, by SuperTheorem, $\vdash_T \bigwedge$vB.

[L to R, if B has v free] Suppose for conditional proof that $\vdash_T \bigwedge$vB. Then by [K4], $\vdash_T$ Bt/v, for each closed term t of T. And since those wff each have <*k* operators, IH tells us that each such wff is a theorem of T iff it is true on *I*. So each such wff is true on *I*. And on *I*, everything in the domain (viz., the domain of closed terms) is assigned as the denotation of some closed term. So the truth of all wff of the form Bt/v on *I* indicates that that $\vdash_T \bigwedge$vB is true on *I* as well.

[R to L, if B has v free] Suppose for conditional proof that $\bigwedge$vB is true on *I*. Then by SuperTruth, B is true also. This means B is satisfied by all sequences on the domain; hence, Bt/v is satisfied by all such sequences, for any closed term t. So all Bt/v is true too, for any closed term t. And since each such Bt/v has <*k* operators, IH tells us that $\vdash_T$ Bt/v iff Bt/v is true on *I*, for any closed term t. So, since T is closed, this means $\vdash_T \bigwedge$vB.

47.2 The *Normal* Model Existence Lemma for $K^=$

47.2: If K is a consistent first order theory with ‘=’, then K has a countable normal model.

Let $K^=$ be any consistent first order theory with ‘=’. Note that $QS^=$ is one such theory, where [$QS^=$1] and [$QS^=$2] are understood as yielding its proper axioms. But $QS^=$ is not the only such theory (and we may speak generally of [$K^=$1] and [$K^=$2] as yielding proper axioms for an arbitrary $K^=$).

By MEL, $K^=$ has a denumerable model M with a domain consisting of the closed terms of Q. We add further constrains on M to ensure a normal model M*.

We can straight-away declare that ‘=’ shall express identity in M*, but this does not secure that every theorem of the form “$c_1 = c_2$” in $K^=$ is true in M*. For the constants c_1 and c_2 are not yet guaranteed to co-refer. We need to rig M* to ensure this.

Here, we define a series of equivalence classes of closed terms (intuitively, the sets of co-referring closed terms). Let c_{j1} be the first closed term in some enumeration of the closed terms of $K^=$. Then, define a set S_1 such that, for any closed term d, d $\in$ S_1 iff c_{j1} = d is a theorem of $K^{=}$ Similarly, where c_{j2} is the second closed term in the enumeration, define S_2 as the set of closed terms d such that $c_{j2} = d$ is a theorem of $K^{=}$. And so on.

Let the domain of M* be the set $\{c_{j1}, c_{j2}, c_{j3}\ldots\}$. (This is countable, since the closed terms *in toto* are denumerable.) Interpret each term in S_1 as denoting c_{j1}, each term in S_2 as denoting c_{j2}, etc. Also, if *s* is any sequence of closed terms $\langle b_1, b_2, b_3\ldots\rangle$ in M, let *s** be the sequence $\langle c_1, c_2, c_3\ldots\rangle$ of closed terms in M* such that for any *n* and *m*, if $c_n = b_m$ is a theorem of $K^{=}$, then c_n in *s** replaces b_m in *s*.

It can then be shown by induction on the number of operators that, for any wff A of $K^{=}$, A is true on M* iff A is true on M. Thus, since M is a model of $K^{=}$, so is M*.

The (Normal) Completeness of $K^{=}$

The Normal MEL can also be used in the Basic Argument for the Completeness of K, to show that any first order theory with '=' (including $QS^{=}$) is complete in relation to its normal models.