

A New Modal Liar¹

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1. Introduction

Standardly, ‘necessarily’ is treated in modal logic as an operator on propositions (much like ‘~’). Some have thought it should be seen instead as a predicate ‘N(x)’ defined on sentences. If that is right, then in lieu of a formula ‘ $\Box p$ ’, one would have something along the lines of ‘ $N(p)$ ’. But the operator view has become standard, mainly because of Montague’s (1968) argument that liar-like paradox results under the predicate view, for systems T and stronger. Moreover, this argument continues to gain adherents (see, e.g., Halbach 2009). However, it has been shown recently that Montague’s argument features a contentious use of the necessitation rule (Dean 2014). Here, I shall join the heterodoxy by arguing that liar-like paradox also results from the operator view (and without the use of the necessitation rule). This, together with Dean (2014), suggests that the advantages of the operator view have been rather exaggerated.

2. Modal lying

The present liar-like paradox consists in the following:

(i) $\sim\Box q$

Assume here that q = the proposition expressed by (i). (Thus, intuitively (i) means “this very proposition is not necessary”). Claim: If the accessibility-relation is reflexive, it can be shown

¹ Thanks to Michael Detlefsen, Volker Halbach, and Alexander Pruss and for helpful feedback on earlier versions of this paper. (At one time, Pruss had (independently) offered a similar argument on his blog, but I am now unable to find the page.)

that q is both true and false, in violation of the law of non-contradiction. The lesson, then, will be that modal systems T and stronger must take special measures to prevent any wff from expressing such a proposition.

In proving this, I shall use the method of modal tableaux as presented in, e.g., Priest (2008) and Girle (2010). (N.B., my presentation is closer to Girle; I find his tableaux more explicit on some points of detail.) One element we shall utilize is, of course, axiom T. Where n is the index for an arbitrary possible world, the axiom says that if $\lceil \Box p \rceil$ is true in world n , infer that $\lceil p \rceil$ is also true in world n :²

Axiom T

$$\begin{array}{ll} \Box p & (n) \\ \therefore p & (n) \end{array}$$

In addition, we will use the more general rule for box-elimination. Namely, if $\lceil \Box p \rceil$ is true in world n , infer that $\lceil p \rceil$ is true in world k , if it has also been established that n has modal access to k (symbolized as “ nAk ”):

$$\begin{array}{ll} \Box\text{-Elim} & \\ \Box p & (n) \\ nAk & \\ \therefore p & (k) \end{array}$$

² Strictly speaking my symbol $\lceil p \rceil$ is ill-formed since, if ‘ p ’ is replaced with a single atomic letter, no concatenation occurs. But suppose the atomic expressions have subject-predicate structure. Then in lieu of ‘ p ’, we could use the compound variable ‘ $\Phi(t)$ ’ within the corner quotes.

Finally, use will be made of the \diamond -Elim rule, which indicates that if $\lceil \diamond p \rceil$ is true in world n , infer that for some world k , nAk is true and infer that $\lceil p \rceil$ is true in k . N.B., this rule is applicable under the proviso that the index k does not appear previously in the proof.

\diamond -Elim

$\diamond p$ (n)

$\therefore nAk$

$\therefore p$ (k)

**where k does occur earlier in the proof.*

Having acknowledged these rules, the proof that q is both true and false can be given as follows:

(Def) $\Box(q \equiv \sim \Box q)$ (n) Definition of q , as per (i)

Argument that q is true:

- (1) $\sim q$ (n) Assume for *reductio*
- (2) $q \equiv \sim \Box q$ (n) From (Def), by Axiom T
- (3) $\Box q$ (n) From (2), (1), by truth-functional logic
- (4) q (n) From (3), by Axiom T

Contradiction at (1) and (4).

Argument that q is false:

- (5) q (n) Assume for *reductio*
- (6) $q \equiv \sim \Box q$ (n) From (Def), by Axiom T
- (7) $\sim \Box q$ (n) From (6), (5), by truth-functional logic

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| (8) $\diamond \sim q$ | (n) From (7), by interdefinability of ‘ \square ’ and ‘ \diamond ’ |
| (9) nAk | From (8), by \diamond -Elim |
| (10) $\sim q$ | (k) From (8), by \diamond -Elim |
| (11) $q \equiv \sim \square q$ | (k) From (Def) and (9), by \square -Elim |
| (12) $\square q$ | (k) From (11), (10), by truth-functional logic |
| (13) q | (k) From (12), by Axiom T |

Contradiction at (10) and (13).

Remark: At no point is the necessitation rule deployed. Thus, Dean’s (2014) misgivings about Montague’s modal liar do not apply here.

One response is to argue that (Def) does not define a legitimate proposition. I am quite willing to agree. Still, modal logic contains no stipulation against something like (Def). So as far as the formalism is concerned, (Def) is perfectly admissible. There may be ways to correct for that, but the point is that it needs correcting.³

3. *Whither Soundness?*

It will likely come as a surprise if systems T and stronger breed paradox. For there are long-established soundness proofs for these systems, i.e., proofs for their consistency. Yet if we can prove that q is both true and false within these systems, they are not consistent. What gives?

³ Post (1970) offers a modal liar akin to the one above. Yet it differs in a crucial way: Post assumes “If a statement entails its negation, then its negation is necessary” (p. 405). He then adds “This generalization does not depend on identifying entailment and strict implication” (ibid). But these statements are puzzling. For if $p \supset \sim p$, it does not follow that $\square \sim p$.

I fear this illustrates how little a soundness proof shows. Remember that if one's premises are inconsistent, a soundness proof is available for any system, by *ex falso quodlibet*. (The point is familiar from Gödel's second incompleteness theorem.) Yet from that sort of "proof," it does not follow there that the formal system is *sound*. The existence of a soundness proof does not suffice for soundness; it must be a soundness proof that starts from consistent premises.

But why think that extant soundness proofs for T and stronger depend on inconsistent premises? Well, either the proposition q (as defined at (Def)) is recognized as atomic or not. If it is atomic, then the proof by induction starts from the impossible base claim that each atomic proposition is consistent. However, suppose q is identified as the nonatomic proposition: $\sim\Box q$. Nevertheless, $\sim\Box q$ would remain equivalent to q ; thus, since the former is inconsistent, so is the latter. In this case too, then, we know there is an inconsistent atomic proposition, which again means the base claim is impossible.

4. Closing Remark

How do we purify a modal system of the modal liar? My suspicion is that it may require substantive revisions, and I dare not go into it here. Yet in one respect, it should be unsurprising that some modal systems generate paradox. As part of the object language ("OL"), the necessity operator ' \Box ' can be adequately defined in the metalanguage as follows:

(ii) $\ulcorner\Box p\urcorner$ is true_{OL} iff $\ulcorner p\urcorner$ is true_{OL} in every possible world.

If ' \Box ' ranges over the same worlds as 'necessarily', (ii) will be exceptionless. Even so, (ii) raises a red flag: Qua definition, it suggests that ' \Box ' is a *semantic* expression in the object language. After all, the right-hand side utilizes the notion of *true-in-OL*. (The point holds, even assuming that 'necessarily true' is not equivalent to 'analytic'.) However, if ' \Box ' is in the object language,

yet is defined partly by ‘true_{OL}’, then absent any further qualifications, Liar-liar like paradox is only to be expected.

Caveat: Some may reject (iii) in favor of something like:

(iii) $\lceil \Box p \rceil$ is true iff, in every possible world, p .

Nonetheless, this really just obscures rather than resolves the matter. Following Kripke (1959; 1963), a “possible world” is defined by a maximally consistent set, and “consistent” is defined in terms of truth. Thus, the *definiens* in (iii) still (covertly) contains a semantic expression; it is equivalent to the condition that $\lceil p \rceil \in S$, for any set S where it is possible for all members to be jointly true_{OL} (and where any consistent superset of $S = S$).

Regardless, I raise this point about ‘true’ only to explain why liar-like paradox in modal logic may be expected. Principles like (ii) or (iii) are not crucial to the derivation of the paradox in section 2. The paradox must be reckoned with regardless.⁴

⁴ Similarly, “modalists” (e.g., Bueno & Shalkowski 2015) may reject principles like (ii) and (iii) on the grounds that ‘ \Box ’ should be taken as primitive. But again, to reject (ii) or (iii) is not to resolve the paradox—these semantic principles were not required in the formal proof of the contradiction.

References

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