# SOUNDNESS AND ITS SIBLINGS

PS is **sound**: If $\Gamma \vdash_{PS} A$, then $\Gamma \vDash_{P} A$. I.e., If A is derivable from Γ in PS, then Γ entails A (where A is a wff in P, and Γ is a [possibly empty] set of wff in P). Subscripts omitted in what follows.

Premises[1]
28.3: If $\vdash A$, then $\vDash A$. [The "Theorem Theorem" for PS]
26.1: If $\Gamma \cup \{A\} \vdash B$, then $\Gamma \vdash A \supset B$. [The Deduction Theorem for PS]

## 28.3: The "Theorem Theorem" for PS

***The Basic Argument***
28.1: The axioms of PS are logically valid.
28.2: The rule of inference in PS (i.e., *modus ponens*) preserves logical validity.
∴ 28.3: If $\vdash A$, then $\vDash A$. [From 28.1 and 28.2]

***Proof of 28.1***
Consider each axiom-schema [PS1]-[PS3] and show that each generates only logically valid wff in P. E.g., consider [PS1], which is "$A \supset (B \supset A)$." There are only four possibilities:

- -The wff replacing both 'A' and 'B' are true.
- -The wff replacing both 'A' and 'B' are false.
- -The wff replacing 'A' is true, and the wff replacing 'B' is false.
- -The wff replacing 'A' is false, and the wff replacing 'B' is true.

By the meaning of '⊃', all of these possibilities result in a truth, no matter what wff are involved. So the axioms generated by [PS1] are all logically valid in P.

- -Argue in the same way with [PS2] and [PS3]

***Proof of 28.2***
Assume that $A \supset B$ is true and A is true. The latter means that the antecedent of $A \supset B$ is true. So by the meaning of '⊃', B must be true too. So *modus ponens* (MP) preserves truth on an interpretation, hence, MP preserves logical validity.

## 28.4: The Soundness Theorem for PS

***The Basic Argument***
Suppose for conditional proof that $\Gamma \vdash A$. If Γ is empty, then the Theorem Theorem ensures that $\Gamma \vDash A$. If Γ is non-empty, then since any derivation is finite, there is a finite set such that $\{A_1 \ldots A_n\} \vdash A$. From this and the Deduction Theorem, it follows that $\vdash A_1 \supset (A_2 \supset (\ldots(A_n \supset A))\ldots)$. So by the Theorem Theorem, $\vDash A_1 \supset (A_2 \supset (\ldots(A_n \supset A))\ldots)$. By the meaning of '⊃', this indicates that any model for $\{A_1 \ldots A_n\}$ is also a model for A. And since $\{A_1 \ldots A_n\} \subseteq \Gamma$, this implies that any model for Γ is also a model for A, i.e., $\Gamma \vDash A$.

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[1] The numbering of propositions follows Hunter.

**26.1: The Deduction Theorem for PS**

Suppose for conditional proof that $\Gamma \cup \{A\} \vdash B$. Then there is a derivation D, a series of wff $D_1 \ldots D_n$, where B is last in the series. Given that, we will show how to build a derivation D* which starts from Γ and ends in $A \supset B$. We do this by (strong) induction on the length of D. This will reveal that no matter how B is derived in D, D* can be built, meaning that $\Gamma \vdash A \supset B$.

Basis:
D is a series of only one wff. Since B must be the final wff in D, this means the derivation consists only in B. Yet D is a derivation starting from $\Gamma \cup \{A\}$; hence, Γ is either empty or is {A} itself and A = B. Also, it must be that B is an axiom. In which case, we can build our derivation D* as follows:

1. B [Axiom]
2. $B \supset (A \supset B)$ [Axiom from PS1]
3. $A \supset B$ [MP from 1, 2]

Inductive Step:
The inductive hypothesis is that, given a derivation D of length $<k$ from $\Gamma \cup \{A\}$ to B, there is a derivation D* from Γ to $A \supset B$. We want to show that if D is of length $k$ $\Gamma \cup \{A\}$ to B, there is still such a derivation D*. Now B has at least one of four possible statuses in the derivation D of length $<k$: (i) B is an axiom, (ii) B is a member of Γ, (iii) B = A, or (iv) B is an immediate consequence (by MP) of two earlier wff in D.

*Case (i) and (ii)*: If B is an axiom, then D* will be as in the basis case. Ditto if B is in Γ, except that line 1 of D* will not be an axiom, but simply assumed as a premise (as a member of Γ).

*Case (iii)*: If B = A, then D is a derivation from $\Gamma \cup \{A\}$ to A, and thus, D* should be a derivation from Γ to $A \supset A$. Yet $A \supset A$ can be derived at any point; see Hunter p. 86 for details.

*Case (iv):* Suppose B is derived by a single application of MP from two earlier wff. If so, then one of these wff must be a conditional, to which the other stands as the antecedent (and B is the consequent). Suppose that the conditional here is $D_i \supset B$, so that the other wff is $D_i$.

Thus, D has $D_i \supset B$ and $D_i$ occurring before B. Now the steps from $\Gamma \cup \{A\}$ to $D_i$ must be less than $k$, given that D has fewer than $k$ wff. Thus, the inductive hypothesis assures us that $\Gamma \vdash A \supset D_i$. Similarly, the length of the derivation of $D_i \supset B$ must be less that $k$, and hence, the inductive hypothesis applies to it too. So we know that $\Gamma \vdash A \supset (D_i \supset B)$.

Now it is also an axiom, by [PS2], that $(A \supset (D_i \supset B)) \supset ((A \supset D_i) \supset (A \supset B))$. So this can be derived in extending D to build D*. By applying MP to this axiom and to $A \supset (D_i \supset B)$ from earlier, we can further extend the derivation to obtain $(A \supset D_i) \supset (A \supset B)$. Since we also have $A \supset D_i$ earlier in the derivation, we can add one more step, applying MP to get $A \supset B$, as desired.