SOUNDNESS OF QS AND QS$^{=}$

QS is **sound**: If $\Gamma \vdash_{QS} A$, then $\Gamma \vDash_{Q} A$. I.e., If A is derivable from Γ in QS, then Γ entails A (where A is a wff in Q, and Γ is a [possibly empty] set of wff in Q). Subscripts omitted below.

Premises
43.5: If $\vdash A$, then $\vDash A$. [The "Theorem Theorem" for QS]
43.1: If $\Gamma \cup \{A\} \vdash B$, then $\Gamma \vdash A \supset B$. [The Deduction Theorem for QS]

43.7: The Soundness Theorem for QS

The Basic Argument
Suppose for conditional proof that $\Gamma \vdash A$. If Γ is empty, then the Theorem Theorem for QS ensures that $\Gamma \vDash A$. If Γ is non-empty, then since any derivation is finite, there is a finite set such that $\{A_1 \ldots A_n\} \vdash A$. From this and the Deduction Theorem for QS, it follows that $\vdash A_1 \supset (A_2 \supset (\ldots(A_n \supset A))\ldots)$. In which case, the Theorem Theorem for QS indicates that $\vDash A_1 \supset (A_2 \supset (\ldots(A_n \supset A))\ldots)$. By the meaning of '⊃', this means that any model for $\{A_1 \ldots A_n\}$ is also a model for A. And since $\{A_1 \ldots A_n\} \subseteq \Gamma$, this implies that any model for Γ is also a model for A, i.e., $\Gamma \vDash A$.

43.1: The Deduction Theorem for QS

We suppose for conditional proof that $\Gamma \cup \{A\} \vdash B$, meaning that there is a derivation D that ends with B. We can then show how to extend D in such a way that it ends with $A \supset B$. We do this by strong induction on the length of D. The strong induction is entirely parallel to that given for the Deduction Theorem for PS. That's because the only features of PS we exploited were:

a. Any wff of the form $A \supset (B \supset A)$ is a theorem.
b. Any wff of the form $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$ is a theorem.
c. MP is the sole rule of inference.

And these are features of QS as well. Plus, the induction did not depend on the metavariables ranging over wff of P vs. Q. So the Deduction Theorem holds for QS.

43.5: The "Theorem Theorem" for QS

The Basic Argument
QAX: The axioms of QS are logically valid.
40.4: The rule of inference in QS (i.e., *modus ponens*) preserves logical validity.
∴ 43.5: If $\vdash A$, then $\vDash A$. [From QAX and 40.4]

Proof of 40.4
The argument here is the same as it was for 28.2. That's because the earlier argument didn't depend on the metavariable ranging over wff of P vs. Q.

Proof of QAX

We consider each axiom-schema [QS1]-[QS7] and show that each generates only logically valid wff in Q. Note first that [QS1]-[QS3] are tautological schemata. And wff from such schemata are always logically valid in Q, for the same reason that they are always logically valid in P (Slight wrinkle: In arguing this, a metavariable in a schema should be seen not as an arbitrary "true wff of P" or "false wff of P," but rather an arbitrary *satisfied* wff of Q, or *unsatisfied* wff of Q.)

Consider [QS6]: $(\bigwedge v(A \supset B) \supset (\bigwedge vA \supset \bigwedge vB))$. Suppose for *reductio* that on an arbitrary interpretation *I*, every sequence satisfies the antecedent (for some wff A and B), but not the consequent. The latter indicates that every sequence satisfies A, but not every sequence satisfies B. Yet that would also suffice for a false antecedent, contrary to supposition. So by *reductio* an arbitrary instance of [QS6] must be true on an arbitrary *I*, i.e., it is logically valid in Q.

Consider [QS5]: $(A \supset \bigwedge vA)$, if v does not occur free in A. Suppose A stands for a wff in Q that is true on an arbitrary *I*. Then, by the definition of truth, every sequence satisfies A on *I*. Now if v does not occur free in A, then either v does not occur at all, or v is bound by '$\bigwedge$'. Either way, adding a v-quantifier to A yields a wff that is still satisfied by all sequences on *I*, hence, $\bigwedge vA$ will be true. Since *I* was arbitrary, this establishes that such a wff is logically valid.

Consider [QS4]: $(\bigwedge vA \supset At/v)$, if t is free for v in At/v. If the antecedent of such a wff is true on an arbitrary *I*, then by the satisfaction-condition for universally quantified wff, every sequence satisfies A on *I*, where v is free in A. Now, if v is replaced with a term t that is *free for v*, there are three possibilities to consider: (i) t is a different variable u, (ii) t is a constant, or (iii) t is a functor f whose free variables, if any, remain free after the replacement of v.

Case (i): The alternate variable u will still be free in A; hence, if A was satisfied before by every sequence on *I*, that still holds when v is switched out with a free u.

Case (ii): At/v is satisfied by every sequence on *I*, since for *some* sequence, t will name the object assigned to the variable v in A.

Case (iii): Either the functor f has or lacks free variables…If f lacks free variables, then Af/v will be satisfied by every sequence on *I*, given that A is satisfied by every sequence and f in this case denotes the object assigned to v in *some* sequence.

If f has free variables, then since A is satisfied by every sequence on *I*, so will Af/v. After all, it won't matter what gets assigned to the free variable(s). For the functor will ultimately denote an object assigned to the free variable v in A from *some* sequence. So Af/v will be satisfied here too, since A was *always* satisfied, no matter what was assigned to v.

So where the antencedent of the wff is satisfied by all sequences on an arbitrary interpretation, the consequent of the wff is too. This establishes that such a wff is logically valid.

Consider [QS7]: Axiom universalization. Any logical validity, by definition, satisfies all sequences on any interpretation. And all axioms from [QS1]-[QS6] have been shown to be logically valid. Hence, by the satisfaction-condition of universally quantified wff, adding '$\bigwedge$' to such axioms will still result in logically valid wff. Ditto with adding universal quantifier to *those* wff, etc. So [QS7] yields only logically valid wff.

Soundness of $QS^=$

The soundness argument for $QS^=$ is the same as for QS, except that we need to show that two more axiom-schemata, [$QS^=1$] and [$QS^=2$], yield only logical validities in Q (where '=' has its normal interpretation). The point is taken to be obvious for [$QS^=1$].

Re: [$QS^=2$], we want to show as logically valid any closure of $x = y \supset (A \supset B)$, where B is like A except y may replace any free x, as long as y is also free wherever it replaces x.

Definition: If A is an open wff in which $v_1 \ldots v_n$ have free occurrences, then A preceded by $\bigwedge v_1 \ldots \bigwedge v_n$ (not necessarily in that order) is a **closure** of A. Whereas, if A is a closed wff, A is a closure of A, and so is $\bigwedge vA$ (where v is any variable).

Imagine that $x = y \supset (A \supset B)$ is not schematic, but rather is an arbitrary example of an open wff of Q, whose closures are at issue. Suppose for conditional proof that some sequence s satisfies the left-hand side on an arbitrary (normal) I. Then, the first two elements of s are identical. Suppose also that the first element satisfies A. And suppose, per [$QS^=2$], that B is just like A except y occurs in at least one position where a free x occurred (and y remains free in that position). Then, the first element of s would equally satisfy B. However, the first element of s is identical to the second element of s. So the second element of s satisfies B as well.

Moreover, s was an arbitrary example of a sequence (which satisfied the antecedent) on an arbitrary (normal) interpretation. So all sequences satisfy it on such an I. Hence, adding universal quantifiers to bind x and y will result in a wff that is still satisfied on any such I, by the satisfaction-condition of the quantifier. And that remains true, no matter how many further quantifiers are added. But these are the closures of the wff with the relevant form. And so, all axioms from [$QS^=2$] are logically valid.